Higher Order Cyclic Homology for Rational Algebras

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Algebraic Topology

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Contents

Introduction

Let k be a commuative ring. Given a k-algebra A and an A-bimodule M, one can define the *Hochschild complex* $HH(A,M) : \Delta^o \to$ k-mod as a certain simplicial k-module, given on an object $[q]$ of Δ^o as $HH(A,M)_q = M \otimes A^{\otimes q}$. The homology of this simplicial module is called the *Hochschild homology of* A *with coefficients in* M and is denoted by $\pi_* HH(A,M)$. In the case $M = A$ we get a *cyclic module* $HH(A) := HH(A,A) : \Lambda^o \to$ k-mod, where Λ is Connes's cyclic category (see e.g. [Lod97] or Section 3.1 below). The *cyclic homology* of this cyclic module is called the *cyclic homology of* A and is written as $HC_*(A)$ (Section 3.1). In fact, the groups $HC_*(A)$ are the homotopy of the homotopy orbits of the realization $|HH(A)|$ under the action of S^1 (Section 3.2). The homotopy fixed points give rise to the *negative cyclic homology of* A, denoted by $HC_*^-(A)$. There also exists a family of related groups, called *periodic Hochschild homology* $HP_*(A)$, and these measure the "difference" between HC_* and HC_*^-, in the sense that there is a long excact sequence (see e.g. [Lod97, Section 5.1])

$$\cdots \to HC_{n-1} \to HC_n^- \to HP_n \to HC_{n-2} \to \cdots .$$

A motivation for studying these homology theories comes from the following relations to K-theory. There are maps $D : K_* \to HH_*$ and $ch : K_* \to HC_*^-$, called the *Dennis trace map* and the *Chern character*, respectively ([Goo86], [Jon87]). There is also a map $\pi : HC_*^- \to H_*$, and the Dennis trace map factors through the other two maps ([Goo86]):

$$\begin{array}{ccc} & HC_*^- & \\ {}^{ch}\nearrow & & \searrow^{\pi} \\ K_* & \xrightarrow{D} & HH_*. \end{array}$$

This can be an aid in calculations of K-theory, since Hochschild and cyclic homology are easier to compute. As an example illustrating this, we quote the following theorem by Goodwillie.

Theorem ([Goo86]). *Let* $A \xrightarrow{f} B$ *be a map of simplicial algebras, where* A *and* B *are rational, such that the kernel* I *in*

$$0 \to I \to \pi_0 A \xrightarrow{\pi_0 f} \pi_0 B \to 0$$

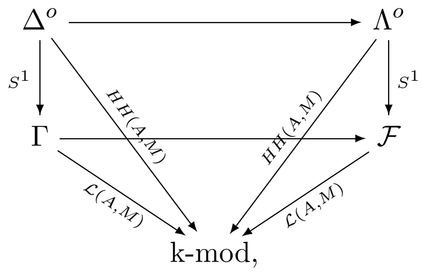
is nilpotent. Then there is a diagram

$$\begin{array}{ccccc} K(A) & \xrightarrow{ch} & HC^-(A) & \longleftarrow & \Sigma HC(A) \\ \downarrow & & \downarrow & & \downarrow \\ K(B) & \xrightarrow{ch} & HC^-(B) & \longleftarrow & \Sigma HC(B), \end{array}$$

where the two squares are homotopy cartesian (in the sense of [GJ99, Section II.8]).

Thus, if we can calculate $HC^-(A)$, $HC^-(B)$, $HC(A)$, and $HC(B)$, we might be able to calculate the K-groups.

Let Γ and $\mathcal{F}$ be the category of finite pointed sets and finite sets, respectively. Loday has observed that when A is commutative, then the Hochschild complex factors through these categories:



where the vertical maps are given by the *simplicial circle* S^1, and the horizontal maps by inclusions. The map $\mathcal{L}(A, M)$ is the *Loday functor.* Hence, we can consider Hochschild homology of A with coefficients in M as the homology associated to the composition

$$\Delta^o \xrightarrow{S^1} \Gamma \xrightarrow{\mathcal{L}(A,M)} \text{k-mod}\,.$$

This construction can be considered for any simplicial finite set X and any functor $F : \Gamma \to$ k-mod. The homology associated to the composition

$$\Delta^o \xrightarrow{L} \Gamma \xrightarrow{F} \text{k-mod}$$

has been studied by Pirashvili in [Pir00b]. Pirashvili focused on the case $L = S^n$ and called the associated homology *n-th order Hochschild homology of F.*

In this thesis we will focus on the case $L = T^n$, where T^n is the simplicial n-torus, and in the case $F = \mathcal{L}(A)$, this gives us the *n-th iterated Hochschild homology of A.*

We will define the n-th order cyclic homology of a functor $F : \mathcal{F} \to$ k-mod as the cyclic homology associated to the n-cyclic module

$$\Lambda^o \times \cdots \times \Lambda^o \xrightarrow{T^n} \mathcal{F} \xrightarrow{F} \text{k-mod},$$

and the *n-th order cyclic homology of a commutative algebra A* as the homology we get in the case $F = \mathcal{L}(A)$. This will correspond to the homotopy orbits of $|HH^{[n]}(A)|$ under the action of the torus T^n (Section 3.2), where $HH^{[n]}(A)$ is the homology associated to $\mathcal{L}(A) \circ T^n$.

There are topological versions of Hochschild and cylic homology, called topological Hochschild homology (THH) and topological cyclic homology (TC). Topological Hochschild homology can be defined by replacing the tensor product by the smash product of spectra in the definition of the Hochschild complex, resulting in a simplicial complex THH in the category of symmetric spectra (see [Bök86] or [BHM93]). More precisely, if k is a commutative symmetric ring spectrum, R a k-algebra, and M a k-symmetric R-bimodule, the q-simplices of $THH(R; M)$ are defined as

$$THH(R; M)_q = M \wedge_k R^{\wedge_k q}.$$

Topological cyclic homology (TC) was defined by Bökstedt, Hsiang, and Madsen in [BHM93] in such a way as to model the S^1-fixed points of THH.

THH and TC are related to K-theory, as well as to the algebraic versions of Hochschild and cyclic homology. In fact, Dundas, Goodwillie, and McCarthy ([DGM]) have shown that there is a diagram

$$\begin{array}{ccccccc} K & \longrightarrow & TC & \longrightarrow & THH^{hS^1} & \longrightarrow & THH \\ & \searrow & & & \downarrow & & \downarrow \\ & & & & HC^- & \longrightarrow & HH. \end{array}$$

These relations can be used to study iterated K-theory. For instance, by using the map $K \to THH$ from the diagram above, we get a map

$$K(K(A)) \longrightarrow THH(THH(A)).$$

In [CD], Carlsson and Dundas study iterated K-theory by considering iterated topological Hochschild homology THH and its associated topological cyclic homology, hopefully capturing the chromatic behaviour of K-theory predicted by Rognes's Red-shift conjecture.

The above map can be continued into iterated Hochschild homology

$$K(K(A)) \longrightarrow THH(THH(A)) \longrightarrow HH(HH(A)),$$

and one of the motivations for this thesis is to shed some light on iterated K-theory by considering iterated Hochschild homology.

Throughout the thesis, k will denote a commutative rational field, and all k-algebras considered will be unital and commutative.

The thesis is organized as follows:

Chapter 1: Hochschild Homology. Section 1.1 contains some preliminaries from simplicial homotopy theory. In Section 1.2, we will define higher order Hochschild homology of functors and algebras and study some simple properties of these constructions. In Section 1.3, we will do some calculations of higher order Hochschild homology in the cases of the polynomial algebra and the truncated polynomial algebra. To obtain these calculations, we will discuss Hochschild homology of differential graded algebras and prove some general results.

Chapter 2: Operations and Decompositions of Hochschild Homology. In Section 2.1, we give a presentation of some results by Pirashvili that will be used to get a decomposition of higher order Hochschild homology. Section 2.2 is devoted to a study of certain important operators on Hochschild homology, namely the so-called *Adams operations*. We will use methods developed by McCarthy ([McC93]) and Bauer ([Bau]) to show how these operators relate to the decomposition obtained in Section 2.1.

Chapter 3: Cyclic Homology. In Section 3.1, we define higher order cyclic homology. Section 3.2 is devoted to showing that higher order cyclic homology corresponds to orbits of Hochschild homology.

Chapter 4: Decomposition of Cyclic Homology. Pirashvili has constructed a decomposition of cyclic homology, and in this chapter we generalize this decomposition to higher order cyclic homology.

Chapter 1

Hochschild Homology

We will consider the homology of the simplicial module we get from compositions

$$\Delta^o \xrightarrow{L} \Gamma \xrightarrow{F} \text{k-mod}$$

in certain special cases. Here Γ is the category of finite pointed sets.

For example,

if $L = S^1$, we get the *Hochschild homology of the functor* F (Definition 1.2.2);

if $L = S^1$ and F is the *Loday functor* $\mathcal{L}(A)$ (A is commutative), we get the classical *Hochschild homology of the algebra* A;

if $L = T^n$ and $F = \mathcal{L}(A)$, we get the n-th iterated Hochschild homology of A (1.2.7).

In Section 1.3, we will calculate the second iterated Hochschild homology of the polynomial ring and of the dual numbers.

1.1 Preliminaries

The main reference for this section is [GJ99].

We begin by defining the *simplicial category* Δ.

Definition 1.1.1. *The category Δ has as objects the ordinal sets $[n] = \{0, 1, ..., n\}$, $n \geq 0$, and the morphisms are the monotonic order-preserving maps. It is generated by the morphisms*

$$\begin{aligned} \delta^i : \quad [n-1] \quad &\to [n], \qquad 0 \leq i \leq n, \\ \sigma^j : \quad [n+1] \quad &\to [n], \qquad 0 \leq j \leq n+1, \end{aligned}$$

defined by

$$\begin{aligned} \delta^i(\{0, 1, ..., n\}) &= \{0, 1, ..., i-1, i+1, ..., n\}, \\ \sigma^i(\{0, 1, ..., n\}) &= \{0, 1, ..., i, i, i+1, ..., n\} \end{aligned}$$

subject to the cosimplicial relations

$$\begin{aligned} \delta^j\delta^i &= \delta^i\delta^{j-1} \quad \text{if } i<j, \\ \sigma^j\sigma^i &= \sigma^i\sigma^{j-1} \quad \text{if } i \le j, \\ \sigma^j\delta^i &= \begin{cases} \delta^i\sigma^{j-1} & \text{if } i<j \\ id \quad \text{if } i=j \text{ or } i=j+1 \\ \delta^{i-1}\sigma^j & \text{if } i>j+1. \end{cases} \end{aligned}$$

Let $\mathcal{C}$ be a category. Functors $X : \Delta^{op} \to \mathcal{C}$ are called *simplicial $\mathcal{C}$-objects*. Here Δ^o denotes the opposite category of Δ. The generating maps of Δ^o are written as δ_i and σ_j. The simplicial $\mathcal{C}$-objects form a category $s\mathcal{C}$, where the morphisms are the natural transformations. If $X \in s\mathcal{C}$ is a simplicial $\mathcal{C}$-object, we write X_n for the image of $[n]$ under X and call it the *n-simplices of X*.

The most important example of these constructions is the category of simplicial objects in the category of sets, denoted by $\mathcal{S}$ and called the category of *simplicial sets*. The basic objects in $\mathcal{S}$ are the *standard simplicial sets*.

Example 1.1.2. The standard simplicial set Δ^n is defined by

$$\Delta^n([k]) = \hom_\Delta([k],[n]).$$

Its non-degenerate k-simplices (that is, the k-simplices that are not in the image of any σ_j's) are given by the injective order-preserving maps $[k] \to [n]$.

The boundary of Δ^n, denoted by $\partial\Delta^n$, is the simplicial set whose non-degenerate k-simplices correspond to the nonidentity injective order-preserving maps $[k] \to [n]$. The *simplicial n-sphere* is the simplicial set defined by $S^n = \Delta(n)/\partial\Delta(n)$.

1.1.1 Simplicial Objects in Abelian Categories

Let $\mathcal{A}$ be an abelian category, and let $A \in s\mathcal{A}$. The *Moore complex* of A has the n-simplices of A as n-chains and boundary defined by

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$

Here d_i is the map induced from the map δ_i in Δ^o. We mostly write C_*A for the Moore complex of A and denote its homology by π_*A. Sometimes, the Moore complex will be denoted simply by A. The notation π_*A makes sense, since the homology of C_*A agrees with the simplicial homotopy of the underlying simplicial set (see e.g. [GJ99, Section III.2]).

Another way we can associate a chain complex to a simplicial simplicial object A in $\mathcal{A}$, is through the *normalized complex* N_*A. The n-chains of this complex are defined as

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A,$$

and the boundaries are given by

$$NA_n \overset{(-1)^n d_n}{\longrightarrow} NA_{n-1}.$$

This is a subcomplex of the Moore complex.

Theorem 1.1.3 ([GJ99, Theorem III.2.5] (Dold-Kan correspondence))**.** *The normalized chain complex N_* gives an equivalence of categories*

$$s\mathcal{A} \to \mathrm{Ch}_+(\mathcal{A}),$$

where $\mathrm{Ch}_+(\mathcal{A})$ is the category of chain complexes in $\mathcal{A}$.

The normalized complex and the Moore complex are connected through the *degenerate subcomplex* D_*A of A, in the sense that $A = N_*A \oplus D_*A$, and it can be shown that D_*A is acyclic. The n-chains of D_*A are given by $DA_n = \sum_0^{n-1} s_i(A_{n-1})$.

Theorem 1.1.4 ([GJ99, Theorem III.2.5])**.** *Let $A \in s\mathcal{A}$. The composition of chain maps*

$$N_*A \xrightarrow{i} A \xrightarrow{p} A/D_*A,$$

where i is the inclusion and p is the canonical projection, is an isomorphism of chain complexes. Moreover, the inclusion $N_*A \to A$ is a natural chain homotopy equivalence.

1.2 Higher Order Hochschild Homology

Hochschild homology of an associative algebra with coefficients in an A-bimodule M is the homology of the complex

$$M \xleftarrow{b} M \otimes A \xleftarrow{b} M \otimes A \otimes A \xleftarrow{b} \cdots,$$

where b is the operator

$$b(a_0, ..., a_n) = \sum_{i=0}^{n} (a_0, ..., a_i a_{i+1}, ..., a_n) + (-1)^n (a_n a_0, ..., a_{n-1}).$$

Here $a_0 \in M$ and $a_i \in A$ for $i \geq 1$. This chain complex is called the *Hochschild complex* and is denoted by $HH(A, M)$. It is the Moore complex of the simplicial module, also denoted by $HH(A, M)$, whose n-simplicies are $HH(A, M)_n = M \otimes A^{\otimes n}$.

The following category is of special interest:

Definition 1.2.1. *The category Γ has as objects all finite pointed sets and as morphisms all basepoint-preserving set maps. A skeleton for this category is given by the objects $n_+ = \{0, 1, ..., n\}$, where 0 is the basepoint.*

(NB: Some authors write Γ^o for what we have called Γ.) We observe that the simplicial circle is an object of $s\Gamma$. Indeed $S^1([n]) = n_+$.

Loday has observed that if A is commutative, then the Hochschild complex factors through the category Γ:

$$\begin{array}{ccc} \Delta^o & & \\ {\scriptstyle S^1}\downarrow & \searrow^{HH(A,M)} & \\ \Gamma & \xrightarrow[\mathcal{L}(A,M)]{} & \text{k-mod}\,. \end{array}$$

Here S^1 is the simplicial circle, and $\mathcal{L}(A, M)$ is the so-called *Loday functor*, which is defined below.

Let $F : \Gamma \to$ k-mod be any functor. Then we get a simplicial k-module by composing with the simplicial circle:

$$\Delta^o \xrightarrow{S^1} \Gamma \xrightarrow{F} \text{k-mod},$$

and, following [Pir00b], we have the following definition:

Definition 1.2.2. *Let F be a functor $\Gamma \to$ k-mod. The Hochschild homology of the functor F is defined as the homology of the Moore complex, written as $HH(F)$, associated to the composition $F \circ S^1 = F(S^1)$.*

This construction works for any simplicial finite set $L : \Delta^o \to \Gamma$, and the complex given by the composition

$$\Delta^o \xrightarrow{L} \Gamma \xrightarrow{F} \text{k-mod}$$

has been studied in [Pir00b]. Pirashvili focused on the case $L = S^n$ and defined n-th order homology of the functor F as the homology we get in this case. We will focus on the case where F is the *Loday functor* (defined below) and X is the n-torus, and this will give us the n-th *iterated Hochschild homology of A*. We will write $HH^{[n]}(F)$ for the Moore complex associated to the composition

$$\Delta^o \xrightarrow{T^n} \Gamma \xrightarrow{F} \text{k-mod}\,.$$

Remark 1.2.3. We consider the n-torus as a simplicial object in the category Γ by using the diagonal map $\Delta^o \to \Delta^o \times \cdots \times \Delta^o$. By the Eilenberg–Zilber theorem, this is the same up to homology, as considering it as an n-simplicial set and then using the total complex (see e.g. [GJ99, Chapter 4]).

The Loday Functor. A functor of special interest is the so-called *Loday functor* $\mathcal{L}(A, M) : \Gamma \to$ k-mod, which is defined as follows (see [Pir00b, Section 1.7]): It takes $\{0, 1, ..., n\}$ to $M \otimes A^{\otimes n}$ and takes a morphism $f : \{0, 1, ..., m\} \to \{0, 1, ..., n\}$ to the morphism f_* given by $f_*(a_0 \otimes \cdots \otimes a_n) = (b_0 \otimes \cdots \otimes b_m)$, where

$$b_j = \prod_{f(i)=j} a_i, \quad j = 0, ..., n,$$

and $b_j = 1$ if $f^{-1}(j) = \emptyset$. Here a_0 is in M. We write $\mathcal{L}(A)$ for $\mathcal{L}(A, A)$. The definition of $\mathcal{L}(A, M)$ can be extended to *commutative differential graded algebras* A, which are defined as follows:

Definition 1.2.4. *A commutative differential graded algebra (CDGA) is a pair (A, δ), where A is a graded commutative algebra; that is, $ab = (-1)^{|a||b|}ba$, and δ is a degree -1-differential $\delta : A \to A$ satisfying (i) $\delta^2 = 0$, and (ii) $\delta(ab) = \delta(a)b + (-1)^{|a|}a\delta(b)$.*

Now let A be a CDGA and let M be a graded differential A-module. Then the Loday functor $\mathcal{L}(A, M)$ is given on objects as above, while a morphism $f : m_+ \to n_+$ is sent to the morphism f_* given by

$$f_*(a_0, ..., a_n) = (-1)^{\epsilon(f,a)}(b_0, b_1, ..., b_m),$$

where the b_j's are as above, and

$$\epsilon(f,a) = \sum_{j=1}^{n-1} |a_j| \left(\sum_{\{k \mid k>j,\ 0\le f(k)\le f(j)\}} |a_k| \right) \tag{1.1}$$

(see [Pir00b, Section 1.7]). The i–th dimensional part of $\mathcal{L}(A,M)$ will be written as $\mathcal{L}_i(A,M)$.

Dually, there is a contravariant functor $\mathcal{J}(C,N) : \Gamma \to$ k-mod, where C is a cocommutative k-coalgebra and N a C-comodule, given on objects by

$$n_+ \mapsto N \otimes C^{\otimes n}.$$

There is also an extension to differential graded coalgebras analogous to the one for $\mathcal{L}(A,M)$. We write $\mathcal{J}(C)$ for $\mathcal{J}(C,C)$, and $\mathcal{J}_i(C,N)$ for the i-th dimensional part of $\mathcal{J}(C,N)$.

The complex obtained from the composition $\mathcal{L}(A,M)\circ L$, where L is a simplicial set, will be denoted by $HH_L(A,M)$. The complex $H_L(A,A)$ will mostly be written as $A^{\otimes L}$. We make the following definition:

Definition 1.2.5. *Let A be a commutative k-algebra and M an A-bimodule. The n-th order Hochschild homology of A with coefficients in M is the homology of the complex $HH_{T^n}(A,M)$.*

Remark 1.2.6. Ordinary Hochschild homology of an algebra A with coefficients in the bimodule M corresponds to the homology of the complex $A^{\otimes S^1}$. We will sometimes write $HH_*(A)$ for this homology.

Lemma 1.2.7. *Let X and Y be sets. The construction $A^{\otimes X}$ is functorial in X, and*

$$A^{\otimes(X\times Y)} \cong (A^{\otimes X})^{\otimes Y}.$$

In particular, higher order Hochschild homology is iterated Hochschild homology.

Proof. We have

$$A^{\otimes X} = \otimes_{x\in X} A,$$

and

$$A^{\otimes X\times Y} = \bigotimes_{(x,y)\in X\times Y} A = \otimes_{y\in Y}\otimes_{x\in X} A = (A^{\otimes X})^{\otimes Y}.$$

□

As an example, we look at the homology of $HH_X(k[G],k)$.

Example 1.2.8. Let G be an abelian group, and let $k[G]$ be its group algebra. We have

$$\mathcal{L}(k[G],k)(n_+) = k \otimes k[G]^{\otimes n} \cong k[G^{\times n}]$$

and $k[G^{\times n}] \cong k[G\otimes\tilde{\mathbb{Z}}[n]]$, where $\tilde{\mathbb{Z}}[n] = \mathbb{Z}[n]/\mathbb{Z}[0]$ is the *free pointed abelian group* on n. This construction is functorial in n_+, so for an arbitrary simplicial $X \in \Gamma$, we get

$$HH_X(k[G],k) \cong k[G\otimes\tilde{\mathbb{Z}}(X)].$$

By definition of reduced homology of X, we have

$$\pi_* HH_X(k[G],k) \cong k[\tilde{H}_*(X;G)].$$

Let $X = S^n$. We get

$$\pi_* HH_{S^n}(k[G], k) \cong k[\tilde{H}_*(S^n, G)].$$

The space $\tilde{\mathbb{Z}}[S^n]$ is the n-th Eilenberg–Mac Lane space $K[\mathbb{Z}, n]$, so $G \otimes \tilde{\mathbb{Z}}[S^n] = K[G, n]$.

Let $X = T^n$. We have $G \otimes \tilde{\mathbb{Z}}[T^2] = K[G \times G, 1] \times K[G, 2]$, so

$$\pi_* HH_{T^2}(k[G], k) \cong k[\tilde{H}_*(T^2; G)] \cong k[\tilde{H}_*(S^1, G \times G)] \otimes k[\tilde{H}_*(S^2, G)].$$

Lemma 1.2.9. *Let A and B be two k-algebras and let X be a simplicial finite set. Then there is an isomorphism*

$$\pi_* HH_X(A \otimes B) \cong \pi_* HH_X(A) \otimes \pi_* HH_X(B).$$

Proof. This is immediate from the Künneth theorem for homology. □

The Hochschild complex has an algebra structure given by the *shuffle product*.

The algebra structure of $HH(A)$. There is a product in the Hochschild complex, called the *shuffle product* $- \times - :: HH(A)_m \otimes HH(A)_n \to HH(A)_{m+n}$, and it is given by

$$(a, a_1, ..., a_p) \times (a', a_{p+1}, ..., a_{p+q}) = \sum_{\sigma} \mathrm{sgn}(\sigma)(aa', a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(p+q)}),$$

where the sum is extended over all (p, q)-*shuffles* σ ([Lod97, Section 4.2]). A (p, q)-shuffle is a permutation σ in Σ_{p+q} such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \text{ and } \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

Lemma 1.2.10 ([Lod97, Lemma 4.2.2]). *The Hochschild boundary b is a graded derivation for the shuffle product, so the Hochschild complex $HH(A)$ is a differential graded algebra.*

1.3 Examples

We calculate the second iterated Hochschild homology of the polynomial ring $k[t]$ and of the truncated polynomial ring $k[\epsilon]/\epsilon^2$. To do this, we develop some general theory for Hochschild homology of commutative differential graded algebras (defined in 1.2.4). The main references for this section are [BV88], [Cn99], and [Lod97, Section 5.4].

1.3.1 Hochschild Homology of DGAs

According to [Lod97, Section 5.4], Hochschild homology of a differential graded algebra (A, δ) is the homology of the bicomplex

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
(A^{\otimes 3})_0 & \xleftarrow{\delta} & (A^{\otimes 3})_1 & \xleftarrow{\delta} & (A^{\otimes 3})_2 & \xleftarrow{\delta} & \\
b\downarrow & & b\downarrow & & b\downarrow & & \\
(A^{\otimes 2})_0 & \xleftarrow{-\delta} & (A^{\otimes 2})_1 & \xleftarrow{-\delta} & (A^{\otimes 2})_2 & \xleftarrow{-\delta} & \\
b\downarrow & & b\downarrow & & b\downarrow & & \\
A_0 & \xleftarrow{\delta} & A_1 & \xleftarrow{\delta} & A_2 & \xleftarrow{\delta} &
\end{array}$$

where

$$(A^{\otimes n+1})_p = \bigoplus_{i_0+\cdots+i_n=p} A_{i_0} \otimes \cdots \otimes A_{i_n}$$

denote the degree p part and

$$\delta(a_0, ..., a_n) = (\delta a_0, a_1, ..., a_n) + \sum_{i=1}^{n} (-1)^{\epsilon_i} (a_0, ..., a_{i-1}, \delta a_i, a_{i+1}, ..., a_n).$$

Here $\epsilon_i = \sum_{j=0}^{i-1} |a_j|$. The Hochschild boundary b is given by

$$b(a_0, ..., a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, ..., a_i a_{i+1}, ..., a_n) + (-1)^e (a_n a_0, ..., a_{n-1}),$$

where $e = |a_n| \sum_{j=0}^{n-1} |a_j|$. The homology of this complex will be denoted $HH_*(A, \delta)$.

Remark 1.3.1. When we talk about the homology of a bicomplex (or a higher complex), we mean the homology of its total complex.

Definition 1.3.2. *The shuffle product in* $HH_*(A, \delta)$*,*

$$\times : HH_p(A, \delta) \otimes HH_q(A, \delta) \to HH_{p+q}(A, \delta),$$

is given by

$$(a, a_1, ..., a_p) \times (a', a_{p+1}, ..., a_{p+q}) = (-1)^e \sum_{\sigma} (-1)^{f(\sigma)} (aa', a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(p+q)}),$$

where the sum is taken over all (p, q)*-shuffles* σ*,*

$$f(\sigma) = \sum_{\{i<p+j \mid \sigma(i)>\sigma(p+j)\}} |a_i||a_j|,$$

and $e = |a'| \sum_{i=1}^{p} |a_i|$*.*

Remark 1.3.3. We see that the sign $(-1)^{f(\sigma)}$ comes from all the pairs (a_i, a_j) such that a_j has been shuffled past a_i, while the sign $(-1)^e$ comes from moving a' past the elements $a_1, ..., a_p$. See [ML63, Section X.12].

Hochschild homology of CDGAs.

Lemma 1.3.4 ([BV88, Proposition 1.1])**.** *For any commutative DGA (CDGA)* (A, δ) *there exists a free algebra* ΛV *and a surjective quasi-isomorphism*

$$(\Lambda V, \delta) \xrightarrow{\sim} (A, \delta).$$

Such a free CDGA is called a *free model of* A. If $(\Lambda V, \delta)$ is a free model for (A, δ), then $HH_*(A, \delta) \cong HH_*(\Lambda V, \delta)$ by [Lod97, Theorem 5.3.5].

The algebra ΛV is the *graded symmetric algebra of* V, with

$$\Lambda V = S\left(\bigoplus_{n\geq 0} V_{2n}\right) \otimes E\left(\bigoplus_{n\geq 0} V_{2n+1}\right),$$

where S and E are the symmetric and exterior algebra functors, respectively.

In [Lod97, Section 5.4], Loday shows that Hochschild homology of a free CDGA $(\Lambda V, \delta)$ is isomorphic to the homology of the total complex $\Omega^*_{(\Lambda V,\delta)}$ of the double complex given by

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
(\Omega^2_{A|k})_0 & \xleftarrow{\delta} & (\Omega^2_{A|k})_1 & \xleftarrow{\delta} & (\Omega^2_{A|k})_2 & \xleftarrow{\delta} & \\
0\downarrow & & 0\downarrow & & 0\downarrow & & \\
(\Omega^1_{A|k})_0 & \xleftarrow{-\delta} & (\Omega^1_{A|k})_1 & \xleftarrow{-\delta} & (\Omega^1_{A|k})_2 & \xleftarrow{-\delta} & \\
0\downarrow & & 0\downarrow & & 0\downarrow & & \\
A_0 & \xleftarrow{\delta} & A_1 & \xleftarrow{\delta} & A_2 & \xleftarrow{\delta} & ,
\end{array}$$

where $A := \Lambda V$. Here $\Omega^n_{A|k}$ is the *graded exterior differential module of* A, defined by

$$\Omega^n_{A|k} = \bar{\wedge}^n_A \Omega^1_{A|k},$$

where $\Omega^1_{A|k}$ is the graded module of Kähler differentials. The symbol $\bar{\wedge}_A$ is the graded exterior product defined on a graded A-module M as $M \bar{\wedge}_A M = M \otimes_A M/\sim$, where the equivalence relation $\sim$ is generated by

$$m \otimes n \sim -(-1)^{|m||n|} n \otimes m \tag{1.2}$$

(see [Lod97, Section 5.4.3]). The graded module of Kähler differentials is generated by the elements da for $a \in A$, and, as in [Lod97], we put $|da| = |a|$ for homogeneous a.

Remark 1.3.5. We remark that with the convention in Formula 1.2, the generators for $\Omega^n_{A|k}$ satisfy

$$dxdy = -(-1)^{|x||y|} dydx.$$

The differential δ in $\Omega^n_{A|k}$ is given by

$$\delta(adv_1 \bar{\wedge} \cdots \bar{\wedge}\, dv_n) = (-1)^n(\delta(a)dv_1 \bar{\wedge} \cdots \bar{\wedge}\, dv_n) + \sum_{i=1}^{n} (-1)^{\epsilon_i}(adv_1 \bar{\wedge} \cdots \bar{\wedge}\, d\delta(v_i) \bar{\wedge} \cdots \bar{\wedge}\, dv_n),$$

where $\epsilon_i = |a| + \sum_{j=1}^{i-1} |v_j|$, as above. The pair $(\Omega^*_{A|k}, \delta)$ is a DGA, whose product, denoted by $\odot$, is defined as

$$(a_0 dv_1 \bar{\wedge} \cdots \bar{\wedge}\, dv_p) \odot (a' dv_{p+1} \bar{\wedge} \cdots \bar{\wedge}\, dv_{p+q}) = (-1)^e a a' dv_1 \bar{\wedge} \cdots \bar{\wedge}\, dv_{p+q},$$

where $e = |a'| \sum_{i=1}^{p} |v_i|$.

Write $A = \Lambda V$ and let $\pi : HH(A, \delta) \to (\Omega^*_{A|k}, \delta)$ be the map defined by

$$\pi_n(a, a_1, ..., a_n) = 1/n!(a, da_1 \bar{\wedge} \cdots \bar{\wedge}\, da_n).$$

Loday's result is the following:

Proposition 1.3.6 ([Lod97, Proposition 5.4.6])**.** *Let $(\Lambda V, \delta)$ be a free commutative DGA and write $A = \Lambda V$. The map $\pi : HH(\Lambda V, \delta) \to (\Omega^*_{A|k}, \delta)$ is a quasi-isomorphism of complexes. In particular,*

$$HH_*(A, \delta) \cong H_*(\Omega^*_{(A,\delta)}).$$

We will show that the map π_* is not just a quasi-isomorphism of complexes, but also a morphism of algebras.

Proposition 1.3.7. *Let $A = \Lambda V$. The map $\pi : (HH(A,\delta), b) \to (\Omega^n_{A|k}, \delta)$ is a morphism of DGAs.*

Proof. We want to show that there is a commutative diagram

$$\begin{array}{ccc} HH(A)_p \otimes HH(A)_q & \xrightarrow{\pi_p \otimes \pi_q} & \Omega^p_A \otimes \Omega^q_A \\ \times \downarrow & & \downarrow \odot \\ HH(A)_{p+q} & \xrightarrow[\pi_{p+q}]{} & \Omega^{p+q}, \end{array}$$

where $\times$ is the shuffle product and $\odot$ is the product in Ω^*_A defined above.

Let $(a, a_1, ..., a_p) \in HH(A)_p$ and $(a', a_{p+1}, ..., a_{p+q}) \in HH(A)_q$. We have

$$(a, a_1, ..., a_p) \times (a', a_{p+1}, ..., a_{p+q}) = (-1)^e \sum_\sigma (-1)^{f(\sigma)} (aa', a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(p+q)}),$$

where the sum is taken over all (m, n)-shuffles σ and the signs are given by

$$f(\sigma) = \sum_{\{i<p+j \mid \sigma(i)>\sigma(p+j)\}} |a_i||a_j|,$$

and $e = |a'| \sum_{i=1}^p |a_i|$. Hence,

$$\begin{aligned} \pi_{p+q}((a, a_1, ..., a_p) \quad &\times \quad (a', a_{p+1}, ..., a_{p+q})) \\ &= \frac{1}{(p+q)!}(-1)^e \sum_\sigma (-1)^{f(\sigma)} (aa', da_{\sigma^{-1}(1)} \bar{\wedge} \cdots \bar{\wedge}\, da_{\sigma^{-1}(p+1)}). \end{aligned}$$

By reordering $(aa', da_{\sigma^{-1}(1)} \bar{\wedge} \cdots \bar{\wedge}\, da_{\sigma^{-1}(p+q)})$ as $(aa', da_1 \bar{\wedge} \cdots \bar{\wedge}\, da_{p+q})$, we introduce exactly the sign $(-1)^{f(\sigma)}$ (see Remark 1.3.3), and therefore, since there are $\binom{p+q}{n}$ shuffles, we see that

$$\begin{aligned} \pi_{p+q}((a, a_1, ..., a_p) \times (a', a_{p+1}, ..., a_{p+q})) &= \frac{1}{(p+q)!}(-1)^e \binom{p+q}{q} (aa', da_1 \bar{\wedge} \cdots \bar{\wedge}\, da_{p+q}) \\ &= (-1)^e \frac{1}{p!q!} (aa', da_1 \bar{\wedge} \cdots \bar{\wedge}\, da_{p+q}). \end{aligned}$$

We have

$$\pi(a, a_1, ..., a_p) \odot \pi(a', a_{p+1}, ..., a_{p+q}) = (-1)^e \frac{1}{p!q!} (aa', da_1 \bar{\wedge} \cdots \bar{\wedge}\, da_{p+q}),$$

so the diagram is commutative, and π is an algebra-map. □

Following [BV88], we let M be the bicomplex $(M_{**}, \delta', 0)$, with $M_{pq} = (\Lambda V \otimes \Lambda^q dV)_p$. The horizontal differential δ' is the unique derivation of degree -1 with $\delta'(a) = \delta(a)$ if $a \in V$ and $\delta'(da) = -d\delta(a)$, and the vertical differential is 0. Write $A = \Lambda V$, and denote the elements of $A \otimes \Lambda^n(dV)$ by $adv_1 \cdots dv_n$. The differential δ' is given on $A \otimes \Lambda^n(dV)$ as

$$\delta'(adv_1 \cdots dv_n) = (-1)^n \delta(a) dv_1 \cdots dv_n + \sum_{i=1}^n (-1)^{g_i} adv_1 \cdots d\delta(v_i) \cdots dv_n,$$

where $g_i = (|a| + 1) + \sum_{j=1}^{i-1} (|v_j| + 1)$.

Proposition 1.3.8. *There is an isomorphism of DGAs*

$$H_* : (A \otimes \Lambda^* dV, \delta') \to (\Omega^*_{A|k}, \delta),$$

given by

$$H_n(adv_1 \cdots dv_n) = (-1)^{h_n} adv_1 \barwedge \cdots \barwedge dv_n,$$

where h_n is the sum of all the $|v_i|$ with odd i: $h_n = \sum_{i\ odd} |v_i|$.

Proof. We first show that the map is well-defined, i.e. that it respects the commutation rules. Let $\sigma \in \Sigma_n$ be a transposition of the elements k and l, where $k \leq l$, in $(1, ..., n)$. We check that the diagram

$$\begin{array}{ccc} A \otimes \Lambda^n dV & \xrightarrow{H_n} & \Omega^n_A \\ \sigma. \downarrow & & \downarrow \sigma. \\ A \otimes \Lambda^n dV & \xrightarrow{H_n} & \Omega^n_A \end{array}$$

commutes, where $\sigma.$ denotes the action of σ.

Let $dv_1 \cdots dv_n \in \Lambda^n dV$. The action of σ is given by

$$\sigma.(dv_1 \cdots dv_n) = (-1)^{e(\sigma)} dv_{\sigma^{-1}(1)}, ..., dv_{\sigma^{-1}(n)},$$

where

$$e(\sigma) = \sum_{\{i<j \,|\, \sigma(i)>\sigma(j)\ i \geq k, j \leq l\}} (|v_i| + 1)(|v_j| + 1).$$

We have

$$H_n(\sigma.(dv_1 \cdots dv_n)) = (-1)^{h'(\sigma)+e(\sigma)} dv_{\sigma^{-1}(1)}, ..., dv_{\sigma^{-1}(n)},$$

where

$$h'(\sigma) = \sum_{\sigma^{-1}(j) \text{ odd}} |v_{\sigma^{-1}(j)}|.$$

Similarly,

$$\sigma.(H(dv_1 \cdots dv_n)) = (-1)^{h_n + e'(\sigma)} dv_{\sigma^{-1}(1)}, ..., dv_{\sigma^{-1}(n)},$$

where

$$e'(\sigma) = \sum_{\{i<j \,|\, \sigma(i)>\sigma(j),\, i \geq k, j \leq l\}} (|v_i||v_j| + 1), \quad \text{and} \quad h_n = \sum_{i \text{ odd}} |v_i|.$$

Thus, we have to show that

$$h'(\sigma) + e(\sigma) \equiv h_n + e'(\sigma) \pmod 2,$$

or, equivalently, that

$$\sum_{\sigma^{-1}(j) \text{ odd}} |v_{\sigma^{-1}(j)}| - \sum_{i \text{ odd}} |v_i| + \sum_{\{i<j \,|\, \sigma(i)>\sigma(j),\, i \geq k, j \leq l\}} (|v_i| + |v_j|) \equiv 0. \tag{1.3}$$

We have

$$\sum_{\sigma^{-1}(j) \text{ odd}} |v_{\sigma^{-1}(j)}| - \sum_{i \text{ odd}} |v_i| = \begin{cases} v_k - v_l & \text{if k even and l odd,} \\ v_l - v_k & \text{if k odd and l even,} \\ 0, & \text{otherwise.} \end{cases}$$

In the first two cases, the elements v_k and v_l cancel with elements coming from the last sum in Formula 1.3, and the remaining terms add up to zero modulo 2. In the remaining two cases, there is an odd number of elements that are jumped over twice to interchange v_k and v_l, so the last sum is zero modulo 2.

Next, we show that the diagram

$$\begin{array}{ccc} A\otimes \Lambda^n dV & \xrightarrow{H_n} & \Omega_A^n \\ {\scriptstyle \delta'}\downarrow & & \downarrow{\scriptstyle \delta} \\ A\otimes \Lambda^n dV & \xrightarrow{H_n} & \Omega_A^n \end{array}$$

commutes. Let $adv_1\cdots dv_n \in A\otimes \Lambda^n dV$. We have

$$\begin{aligned} \delta(H_n(adv_1\cdots dv_n)) &= (-1)^{h_n}(\delta(adv_1\,\bar\wedge\cdots\bar\wedge\, dv_n)) \\ &= (-1)^{h_n}(-1)^n\delta adv_1\cdots dv_n \\ &\quad +\sum_{i=1}^{n}(-1)^{h_n+\epsilon_i}adv_1\,\bar\wedge\cdots\bar\wedge\, d\delta v_i\,\bar\wedge\cdots\bar\wedge\, dv_n. \end{aligned}$$

Similarly,

$$\begin{aligned} H_n(\delta'(adv_1\cdots dv_n)) &= H_n(\delta'(adv_1\cdots dv_n)) \\ &= (-1)^{h_n}(-1)^n\delta adv_1\cdots dv_n \\ &\quad +\sum_{i=1}^{n}(-1)^{i-1}(-1)^{h_n+g_i}adv_1\cdots d\delta v_i\cdots dv_n. \end{aligned}$$

Since $h_n+\epsilon_i=(i-1)+h_n+g_i$, the diagram commutes. □

Corollary 1.3.9. *Let V be a graded module. There is an isomorphism of graded algebras*

$$\pi_*HH(HH(\Lambda V,\delta))\to H_*(\Lambda(V\oplus d_1V\oplus d_2V\oplus d_2d_1V),\delta''),$$

where δ'' is given by $\delta''(a)=\delta(a)$, $\delta''(d_1a)=-d_1\delta(a)$ $\delta''(d_2a)=-d_2\delta(a)$ and $\delta''(d_2d_1a)=d_2d_1\delta(a)$.

Proof. From Proposition 1.3.6 and Proposition 1.3.7, we have that there is a quasi-isomorphism of algebras

$$\pi : HH(\Lambda V,\delta)\xrightarrow{\sim}(\Omega_A^*,\delta),$$

and by Proposition 1.3.8 there is an isomorphism of DGAs $(\Omega^n_{A|k},\delta)\cong(A\otimes\Lambda^n dV,\delta')$. Hence, there is a quasi-isomorphism

$$HH(HH(\Lambda V,\delta))\xrightarrow{\sim}HH(\Lambda(V\oplus d_1V),\delta'),$$

but, by Proposition 1.3.6 and Proposition 1.3.8 again, there is a quasi-isomorphism

$$HH(\Lambda(V\oplus d_1V),\delta')\xrightarrow{\sim}(\Lambda((V\oplus d_1V)\oplus(d_2V\oplus d_2d_1V)),\delta''),$$

where the differential δ'' is given by

$$\begin{aligned}\delta''(a) &= \delta'(a) = \delta(a),\\ \delta''(d_1a) &= \delta'(d_1a) = -d_1\delta(a),\\ \delta''(d_2a) &= -d_2\delta'(a) = -d_2\delta(a),\\ \delta''(d_2d_1a) &= -d_2\delta'(d_1a) = d_2d_1\delta(a)\end{aligned}$$

□

This result will be used to calculate the second iterated Hochschild homology of some CDGAs.

1.3.2 Example 1

Let $A = k[t]$, with $|t| = 0$. Then $A = \Lambda V$, where $V = kt$ is the free module of rank 1 and $\delta = 0$. By Corollary 1.3.9,

$$\pi_* HH(HH(A,\delta)) \cong H_*(\Lambda(V \oplus d_1V \oplus d_2V \oplus d_2d_1V, \delta''),$$

and from the description of the differential given in the corollary, we see that $\delta'' = 0$. Thus

$$\pi_* HH(HH(A)) \cong \Lambda V \otimes \Lambda d_1V \otimes \Lambda d_2V \otimes \Lambda d_2d_1V.$$

If we put $\epsilon = d_1t$, $\eta = d_2t$, and $x = d_2d_1t$, then we can write

$$\pi_* k[t]^{\otimes T^2} \cong \pi_* HH(HH(k[t])) \cong k[t] \otimes k[\epsilon]/\epsilon^2 \otimes k[\eta]/\eta^2 \otimes k[x] \cong k[t,\epsilon,\eta,x]/\epsilon^2 = \eta^2 = 0,$$

where $|t| = 0$, $|\epsilon| = 1$, $|\eta| = 1$, and $|x| = 2$.

Remark 1.3.10. There is an isomorphism

$$k[t]^{\otimes X} \cong \bigoplus_{n\geq 0} t^n k[\mathrm{Sp}^n(X)],$$

where $Sp^n(X)$ denotes the n-th symmetric product of X, i.e. the n-fold product of X divided out by the action of the symmetric group.

By [Mil69], the homology of $Sp^n(T^2)$ is given by

$$H_*(Sp^n(T^2)) \cong H_*T^2 \otimes H_*(\mathbb{C}P^{n-1}) \cong k[\epsilon]/\epsilon^2 \otimes k[\eta]/\eta^2 \otimes k[\xi]/\xi^n,$$

where ϵ and η have degree 1 and ξ has degree 2. Hence, we can write

$$\pi_*(k[t])^{T^2} \cong H_*T^2 \otimes (\oplus t^n k[\xi]/\xi^n).$$

But $\oplus t^n k[\xi]/\xi^n \cong k[t, t\xi]$, so if we let $x = t\xi$, we get

$$\pi_*(k[t])^{\otimes T^2} \cong k[t,\epsilon,\eta,x]/\epsilon^2 = \eta^2 = 0,$$

just as above.

1.3.3 Example 2

We calculate the second iterated Hochschild homology of the truncated polynomial algebra $A = k[\epsilon]/\epsilon^2$. By [Lod97, 5.4.14], A admits the free model $(\Lambda V, \delta)$ where $V = kx \oplus ky$, $\delta(x) = 0$, $\delta(y) = x^2$, $|x| = 0$, and $|y| = 1$, and, by Lemma 1.3.9,

$$\pi_* HH(HH(\Lambda V, \delta)) \cong H_*(\Lambda(V \oplus d_1 V \oplus d_2 V \oplus d_2 d_1 V), \delta'').$$

The generators for $\Lambda(V \oplus dV \oplus dV \oplus d^2 V)$ are x, $d_1 x$, $d_2 x$, $d_2 d_1 x$, y, $d_1 y$, $d_2 y$ and $d_2 d_1 y$, and $d^2 y$, and we have $d_1 x^2 = d_2 x^2 = y^2 = d_2 d_1 y^2 = 0$.

Using the description of the differential given in Lemma 1.3.9 , we get that

$$\delta''(x) = 0, \quad \delta''(y) = x^2, \quad \delta''(d_1 x) = \delta''(d_2 x) = 0,$$

$$\delta''(d_1 y) = -2x d_1 x, \quad \delta''(d_2 y) = -2x d_2 x,$$

$$\delta''(d_2 d_1 x) = 0, \quad \delta''(d_2 d_1 y) = 2x d_2 d_1 x,$$

In the following table, the entries in the second column are the generators, and the entries on the right are their images under δ''.

$(\mathrm{I})_j$	$(dy)^j$	$-2jx(dx)(dy)^{j-1}$
$(\mathrm{II})_j$	$y(dy)^j$	$x^2(dy)^j + 2jyx(dx)(dy)^{j-1}$
$(\mathrm{III})_j$	$(dy)^j(d^2y)$	$-2jx(dx)(dy)^{j-1}d^2y + 2x(dy)^j(d^2x)$
$(\mathrm{IV})_j$	$y(dy)^j(d^2y)$	$x^2(dy)^j(d^2y) + 2jyx(dx)(dy)^{j-1}(d^2y) - 2xy(dy)^j(d^2x)$

For convenience, we write down the following table,

$dx(\mathrm{I})_j$	$(dx)(dy)^j$	0
$dx(\mathrm{II})_j$	$(dx)y(dy)^j$	$x^2(dx)(dy)^j$
$dx(\mathrm{III})_j$	$(dx)(dy)^j(d^2y)$	$2x(dx)(dy)^j(d^2x)$
$dx(\mathrm{IV})_j$	$(dx)y(dy)^j(d^2y)$	$x^2(dx)(dy)^j(d^2y) - 2x(dx)y(dy)^j(d^2x)$,

which shows the elements of the previous table after multiplication by dx.

One can check that the only elements of $\ker(\delta'')$ are, up to multiples by x and d^2x, of the form

$(1)_j$	$dx(\mathrm{I})_j$
(2)	$x(\mathrm{III})_0 - 2(\mathrm{II})_0(d^2x)$
$(3)_j$	$2(j+1)(dx)(\mathrm{II})_j - x(\mathrm{I})_{j+1}$
$(4)_j$	$(j+1)(dx)(\mathrm{III})_j - (\mathrm{I})_{j+1}(d^2x)$.

Remark 1.3.11. A systematic way to check which elements can be in the kernel of δ'' is to consider the elements y, ydy, and d^2y in turn.

All these elements are in the image of δ'' after multiplication by x. Indeed,

$$\begin{aligned} x \cdot (1)_j &= -1/(2(j+1))\delta''(\mathrm{I}_{j+1}), \\ x \cdot (2)_j &= \delta''(\mathrm{IV}_0), \\ x \cdot (3)_j &= -\delta''(\mathrm{II}_{j+1}), \\ x \cdot (4)_j &= -1/2\delta''(\mathrm{III}_{j+1}). \end{aligned}$$

Moreover, from the first table we see that all elements in the image of δ'' have an x-factor, and the only elements in the kernel, and not in the image, are of the form

$a_j = (1)_j = dx(dy)^j,$

$b = (2) = xd^2y - 2yd^2x,$

$c_j = (3)_j = 2(j+1)dxy(dy)^j - x(dy)^{j+1},$

$d_j = (j+1)dx(dy)^j d^2y - (dy)^{j+1}d^2x.$

Hence,

$$\pi_* HH(HH(k[x]/x^2])) \cong H_*(\Lambda(V \oplus d_1V \oplus d_2V \oplus d_2d_1V), \delta'') \cong k[d^2x]\langle a_j, b, c_j, d_j\rangle.$$

That is, the second order Hochschild homology of $k[\epsilon]/\epsilon^2$ is a $k[d^2x]$-module on the generators a_j, b, c_j, and d_j. The degrees are $|a_j| = 2j+1$, $|b| = 3$, $|c_j| = 2j+2$, and $|d_j| = 2j+4$.

Remark 1.3.12. We remark that

$$\begin{array}{rcl} a_ja_k &=& 0, \\ b^2 &=& 0, \\ c_jc_k &=& 0 \ \ \text{(it's in the boundary)}, \\ d_jd_k &=& (j-k)dxd^2y(dy)^{j+k+1}d^2x, \\ a_jc_k &=& xa_{j+k+1} = 0, \\ a_jd_k &=& -a_{j+k+1}d^2x, \\ bc_j &=& 0 \ \ \text{(in the boundary)}, \end{array}$$

for all j and k.

Chapter 2

Operations and Decompositions of Hochschild Homology

There is a natural decomposition

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F),$$

which in the case $F = \mathcal{L}(A)$ is known as the *Hodge decomposition* of Hochschild homology. It was obtained by Loday ([Lod89]), and independently by Gerstenhaber and Schack ([GS87]), by giving the Hochschild complex a *Hopf-algebra* structure (that is, a structure of a bialgebra with certain relations between its multiplication and comultiplication – see e.g. [Lod97, Appendix A.2]), and using this structure to define some orthogonal idempotents $e_n^{(i)}$ in $\mathbb{Q}[\Sigma_n]$, called the *eulerian idempotents*. The eulerian idempotents were shown to commute with the Hochschild boundary

$$be_n^{(i)} = e_{n-1}^{(i)} b, \text{ if } n \geq 1, \text{ and } 1 \leq i \leq n,$$

so the Hochschild complex $HH(F)$ splits into a direct sum of complexes

$$HH_n^{(i)}(F) := e_n^{(i)} HH_n(F).$$

The decomposition follows by taking homology.

By using the eulerian idempotents, one can define operations on the Hochschild complex, called λ-*operations*, or *Adams operations*, by

$$\bar{\lambda}_n^k := ke_n^{(1)} + \cdots + k^n e_n^{(i)}, \quad n \geq 1.$$

Later, McCarthy ([McC93]) gave a geometric interpretation of these operations by using edgewise subdivision of a simplicial set and the r-fold covering maps of the circle.

In [Pir00b], Pirashvili gave a generalization of the decomposition of Loday by a completely different route, involving the construction of a spectral sequence abutting to $\pi_* F(X)$. Here $F : \Gamma \to$ k-mod is any functor and X any pointed simplicial set. He showed that in the special case where X is S^n, the spectral sequence degenerates, and that in the case $n = 1$ the resulting decomposition corresponds to Loday's decomposition. In the first section of this chapter we will give a presentation of these results and use his methods to get a decomposition of $\pi_* F(T^n)$.

Using methods similar to McCarthy's, Bauer ([Bau]) was able to further generalize this result. More precisely, by showing that $A^{\otimes S^1\wedge X}$ is a Hopf algebra, she got a decomposition of $\pi_n A^{\otimes S^1\wedge X}$. We will look at her methods in Section 2.2 and use them to define the Adams operations on higher order Hochschild homology.

2.1 Pirashvili's Decomposition of Hochschild Homology

As before, Γ is the category of finite pointed sets and basepoint-preserving maps. Following [Pir00b], we let Γ-mod denote the category of all covariant functors from Γ to the category of k-modules and let mod $-\Gamma$ denote the category of all contravariant functors. We write $\Gamma^n := k[\mathrm{Hom}_\Gamma(-, n_+)]$ and $\Gamma_n := k[\mathrm{Hom}_\Gamma(n_+, -)$. These are, for $n \geq 0$, projective generators of the categories $\Gamma -$mod and mod $-\Gamma$, respectively. There is a bifunctor $-\otimes_\Gamma - : (\text{mod} -\Gamma) \times (\Gamma - \text{mod}) \to$ k-mod such that

$$\begin{aligned} F(n_+) &= F(-) \otimes_\Gamma \Gamma_n \\ F(n_+) &= \Gamma^n \otimes_\Gamma F(-). \end{aligned}$$

Moreover, $- \otimes_\Gamma -$ is a left balanced bifunctor (in the sense of [Wei94]), so its left derived functors with respect to each variable are isomorphic. We denote this common value by $\mathrm{Tor}^\Gamma_*(-,-)$.

Remark 2.1.1. This construction works with any small category $\mathcal{C}$ in place of Γ. That is, for any small category $\mathcal{C}$ one can define the categories $\mathcal{C} -$ mod and mod $-\mathcal{C}$, the tensor product $-\otimes_{\mathcal{C}} -$, and the derived functor $\mathrm{Tor}^{\mathcal{C}}_*(-,-)$, all defined in an analogous way to the case $\mathcal{C} = \Gamma$ above. See [PR02].

The basic spectral sequence, from which we get all the decomposition results, is given by the following proposition.

Proposition 2.1.2 ([Pir00b, Proposition 1.6])**.** *Let F be an object of $\Gamma -$mod and let C_* be a nonnegative chain complex of Γ-modules, whose components are projective objects of mod $-\Gamma$. Then there exists a first quadrant spectral sequence*

$$E^2_{pq} = \mathrm{Tor}^\Gamma_p(H_q(C_*), F) \Rightarrow H_{p+q}(C_* \otimes_\Gamma F).$$

Suppose

$$\mathrm{Ext}^{m-n+1}_{\text{mod} -\Gamma}(H_n(C_*), H_m(C_*)) = 0, \quad \text{for} \quad n < m.$$

Then the spectral sequence is degenerate at E^2, and one has the decomposition

$$H_n(C_* \otimes_\Gamma F) \cong \bigoplus_{p+q=n} \mathrm{Tor}^\Gamma_p(H_q(C_*), F),$$

which is natural in F.

Remark 2.1.3. There is a completely analogous result where Γ is replaced by the category of all finite sets $\mathcal{F}$ (see [Pir00b, Section 3.2]). This version will be used in Chapter 4, where we look at a decomposition of cyclic homology.

By using this result, one can construct a spectral sequence converging to the homology of $F(X)$, where F is as before and X is a based simplicial set.

Theorem 2.1.4 ([Pir00b, Theorem 2.4]). *Let F be a left Γ-module and let X be a based simplicial finite set. Then there exists a spectral sequence*

$$E^2_{pq} = \operatorname{Tor}^{\Gamma}_p(\mathcal{J}_q(H_*X), F) \Longrightarrow \pi_{p+q}(F(X)),$$

where $F(X)$ is the composition

$$\Delta^o \xrightarrow{X} \Gamma \xrightarrow{F} K - Mod.$$

Moreover, any simplicial map $X \to X'$ induces an isomorphism $\pi_*(F(X)) \to \pi_*(F(X'))$ as soon as $H_*(X) \to H_*(X')$ is an isomorphism.

Pirashvili proved this result by constructing a chain-complex Γ_X whose homology is given by $H_i(\Gamma_X) \cong \mathcal{J}_i(H_*X)$, in such a way that $F(X) \cong \Gamma_X \otimes_\Gamma F$. By using $C_* = \Gamma_X$ in Proposition 2.1.2, he obtained the result.

The following objects of Γ-mod are of special importance.

Definition 2.1.5. *(a) Let t be the contravariant functor $t : \Gamma \to$ k-mod that is given on objects by*

$$t(n_+) = \hom_{Sets_*}(n_+, k),$$

where $Sets_*$ is the category of pointed sets. It is given on morphisms by precomposition.

(b) The Γ-module θ is given on an object n_+ as the dual of the module

$$(\theta^n)^*(n_+) = k\{S \subset n_+ | |S| \leq n\}/k(\{S \subset n_+ | 0 \in S\} \cup \{S \subset n_+ | |S| < n\}).$$

A morphism $f : m_+ \to n_+$ induces a morphism given by

$$f_*(S) = \begin{cases} 0, & \text{if } 0 \in f(S) \text{ or } |f(S)| < n \\ f(S), & \text{otherwise.} \end{cases}$$

In [Pir00b, Section 1.8], Pirashvili calculated $\mathcal{L}(k[x]/x^2)$, where the generator x has degree $d > 0$. We repeat the argument for odd d here.

Lemma 2.1.6 ([Pir00b, Example 1.8]). *Let $|x| = d$, where $d > 0$. We have*

$$\mathcal{L}_i(k[x]/x^2) \cong \begin{cases} \Lambda^j t^* & \text{if } i = jd \text{ and } d \text{ is odd} \\ (\theta^j)^* & \text{if } i = jd \text{ and } d \text{ is even} \\ 0 & \text{if } i \neq jd \end{cases},$$

where Λ^i is the exterior product and F^* is the dual of the functor F.

Proof. Let $A = k[x]/x^2$. We give Pirashvili's proof of this result in the case where d is odd. We first prove that $\mathcal{L}_{jd}(A)(n_+)$ and $\Lambda^j t^*(n_+)$ are isomorphic. We have

$$\mathcal{L}(A)(n_+) = (k \oplus kx)^{\otimes n} = \bigoplus_{0 \leq j \leq n} (kx^{\otimes j})^{\binom{n}{j}},$$

so $\mathcal{L}_i(A) = 0$ if $i \neq jd$, and

$$\mathcal{L}_{jd}(A)(n_+) \cong k^{\binom{n}{j}}.$$

A basis for $\mathcal{L}_{jd}(A)(n_+)$ is given by the elements $x_J = (x_1, ..., x_n)$, where J runs over the set of j-element subsets of $\{1, ..., n\}$ and

$$x_i = \begin{cases} x, \text{ if } i \in J \\ 1, \text{ if } i \neq J, \end{cases} .$$

There is an isomorphism

$$\Lambda^j t^*(n_+) \cong \mathcal{L}_{jd}(A)(n_+),$$

given by sending a j-element subset J of $\{1, ..., n\}$ to x_J.

Now let $f : m_+ \to n_+$ be a morphism in Γ. On the basis elements of $\mathcal{L}_{jd}(A)(n_+)$ we have

$$f_*(x_1, ..., x_m) = (-1)^{\epsilon(f,x)}(b_1, ..., b_n),$$

where $b_j = \prod_{j=f(i)} x_i$. Since $x^2 = 0$, we get that

$$f_*(x_1, ..., x_n) = \begin{cases} 0, \text{ if } |f(J)| < j \\ x_{f(J)}, \text{ if } |f(J)| = j. \end{cases}$$

The sign is given by Formula 1.1 above, i.e.

$$\epsilon(f, x) = \sum_{j=1}^{n-1} |x_j| \left(\sum_{\{k \,|\, k>j,\ 0 \leq f(k) \leq f(j)\}} |x_k| \right).$$

Hence,

$$\epsilon(f, x) = \Sigma_{l \in J} d^2 p,$$

where $p = \#\{k > l | f(k) \leq f(l)\}$. We have to show that this sign agrees with the one in the exterior power.

The module $\Lambda^j t^*(n_+)$ is generated by objects of the form $i_1 \wedge i_2 \wedge \cdots \wedge i_j$, where the i_k's are ordered and $i_j \leq n$. The map induced by an $f : m_+ \to n_+$ is given on these generators by

$$f_*(i_1 \wedge \cdots \wedge i_j) = (-1)^{\epsilon}(f(i_{k_1}) \wedge f(i_{k_2}) \wedge \cdots \wedge f(i_{k_j})),$$

where the ϵ in the sign is the number of transpositions needed to order the terms in ascending order. Since $\epsilon(f, a) = d^2\epsilon$, the sign agrees with the one above. □

Since the dual of the algebra $H_*(S^d)$ can be written as $\cong k[x]/x^2$, where $|x| = d$, we get

$$\mathcal{J}_i(H_* S^d) \cong \begin{cases} \Lambda^j t \text{ if } i = jd \text{ and } d \text{ is odd} \\ \theta^j \text{ if } i = jd \text{ and } d \text{ is even} \\ 0 \text{ if } i \neq jd \end{cases}$$

by dualizing the above result.

The following proposition gives a decomposition of $\pi_* A^{\otimes S^d}$.

Proposition 2.1.7 ([Pir00b, Corollary 2.5]). *Let $F \in \Gamma$-mod. Then there exists a spectral sequence*

$$E^2_{pq} \Rightarrow \pi_{p+q} F(S^d), \quad d \geq 1,$$

with $E^2_{pq} = 0$ if $q \neq dj$

$$E^2_{pq} = \mathrm{Tor}^\Gamma_p(\Lambda^j t, F) \text{ if } q = dj \text{ and } d \text{ is odd, and,}$$

$$E^2_{pq} = \mathrm{Tor}^\Gamma_p(\theta^j, F) \text{ if } q = dj \text{ and } d \text{ is even,}$$

Moreover, the spectral sequence degenerates:

$$\pi_n F(S^d) \cong \bigoplus_{p+dj=n} \mathrm{Tor}^\Gamma_p(\Lambda^j t, F)$$

if d is odd, and

$$\pi_n F(S^d) \cong \bigoplus_{p+dj=n} \mathrm{Tor}^\Gamma_p(\theta^j, F)$$

if d is even.

The existence of this spectral sequence follows from Theorem 2.1.4 and the calculations done in Lemma 2.1.6. The fact that the sequence degenerates is shown by checking that the extension criterion in Proposition 2.1.2 is satisfied. In the cases involving $\Lambda^i t$, this is easy, because $\Lambda^i t$ is injective for all i ([Pir00b, Section 1.4]).

Pirashvili has shown that the decomposition for $d = 1$ coincides with Loday's decomposition in [Lod89],

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H^{(i)}_n(F)$$

for every left Γ-module. That is, for any $i \geq 1$, one has a natural isomorphism

$$H^{(i)}_n(F) \cong \mathrm{Tor}^\Gamma_{n-i}(\Lambda^i t, F), \quad n \geq i.$$

In fact, in [Pir00b, Theorem 2.7] Pirashvili gives an axiomatic characterization of the decomposition of $\pi_n F(S^1)$ by showing that any natural decomposition having certain properties is of the form in Proposition 2.1.7.

By using Theorem 2.1.4 in the case where X is the n-torus, we get a decomposition of $\pi_* F(T^n)$, and hence of iterated Hochschild homology.

Corollary 2.1.8. *There exists a spectral sequence*

$$E^2_{pq} = \bigoplus_{m_1+m_2+\cdots+m_n=q} \mathrm{Tor}^\Gamma_p(\Lambda^{m_1} t \otimes \Lambda^{m_2} t \otimes \cdots \otimes \Lambda^{m_n} t, F) \Longrightarrow \pi_{p+q} F(T^n).$$

Moreover, the spectral sequence degenerates:

$$\pi_r F(T^n) \cong \bigoplus_{m_1+m_2+\cdots+m_n+p=r} \mathrm{Tor}^\Gamma_p(\Lambda^{m_1} t \otimes \Lambda^{m_2} t \cdots \otimes \Lambda^{m_n} t, F).$$

Proof. We have

$$H_*(T^n) \cong H_*(S^1) \otimes \cdots \otimes H_*(S^1),$$

and for any cocommutative k-coalgebra C and C', we have $\mathcal{J}_q(C \otimes C') \cong \oplus_{m+n=q} \mathcal{J}_m(C) \otimes \mathcal{J}_n(C')$. Thus,

$$\mathcal{J}_q(H_*(T^n)) \cong \bigoplus_{m_1+\cdots m_n=q} \left(\mathcal{J}_{m_1}(H_*S^1) \otimes \cdots \otimes \mathcal{J}_{m_n}(H_*S^1)\right).$$

The existence of the spectral sequence follows from Theorem 2.1.4 and the calculation in Lemma 2.1.6.

The fact that the spectral sequence degenerates follows from Proposition 2.1.2, because $\Lambda^i t$ is injective, and hence $\otimes_1^n \Lambda^{m_i} t$ is injective ([Pir00b, Section 1.3]) and

$$\mathrm{Ext}^j_{\mathrm{mod}-\Gamma}(-, \otimes_1^n \Lambda^{m_i} t) = 0$$

for $j > 0$ ([Pir00b, Section 1.3]). □

2.1.1 Smooth Functors

The concept of smoothness of functors is explored in [Pir00b, Section 4.3], where the following theorem, called the *Hochschild–Kostant–Rosenberg theorem for functors* (HKR-theorem), can be found.

Theorem 2.1.9 ([Pir00b, Theorem 4.6] (HKR)). *Let F be a smooth left Γ-module, and let X be a connected pointed simplicial set. Then*

$$\pi_k F(X) \cong \mathcal{J}_k(H_*X) \otimes_\Gamma F.$$

The following lemma enables us to use this theorem for Hochschild homology of smooth algebras A.

Lemma 2.1.10 ([Pir00b, Lemma 4.5]). *Let A be a smooth commutative algebra of finite type. Then $\mathcal{L}(A)$ is a smooth functor.*

Hence, if A is a finitely generated smooth algebra, then the HKR-theorem implies that

$$\pi_k A^{\otimes T^n} = \left(\bigoplus_{m_1+\cdots+m_n=k} \Lambda^{m_1} t \otimes \cdots \otimes \Lambda^{m_n} t \right) \otimes_\Gamma \mathcal{L}(A).$$

In the case $n = 1$, this description, together with the concept of *cross effect for functors*, can be used to show that

$$\pi_k A^{\otimes S^n} \cong \Omega^n_A, \tag{2.1}$$

where Ω^n_A are the Kähler differentials (see [Pir00b, Proposition 1.15].) We will display some of the ingredients that go into this result.

Cross effect. As in [Pir00b], we let Ω denote the category of finite sets $\langle n\rangle := \{0, 1, ..., n\}$ and surjections. According to Remark 2.1.1, one can define the categories $\mathrm{mod}-\Omega$ and $\Omega-\mathrm{mod}$, and the tensor product $-\otimes_\Omega --$ just as for the category Γ.

In [Pir00a], Pirashvili has shown that there is an equivalence between the categories $\Gamma-\mathrm{mod}$ and $\Omega-\mathrm{mod}$, and between $\mathrm{mod}-\Gamma$ and $\mathrm{mod}-\Omega$, given by the so-called *cross–effect functors*. This allows calculations to be done in the much smaller category Ω rather than in the category Γ.

We sketch the construction of the cross–effect functors here. For details, see [Pir00b, Section 1.10].

Let $n \geq 1$ and $1 \leq i \leq n$, and let r_i be the map $r_i(n_+) \to (n-1)_+$ defined by

$$r_i(j) = \begin{cases} 0, & \text{if } j = i, \\ j, & \text{if } j < i, \\ j-1, & \text{if } j > i. \end{cases}$$

Now let T be an object of $\Gamma-\mathrm{mod}$. Then the *cross effect* of T is the functor $\mathrm{cr}(T)$ in $\Omega-\mathrm{mod}$ defined on objects by

$$\mathrm{cr}(T)(\langle n\rangle) = \ker(r_* : T(n_+) \to \prod_{i=1}^{n} T((n-1)_+)).$$

Similarly, if F is in $\mathrm{mod}-\Gamma$, then there is an object of $\mathrm{mod}-\Omega$ defined on objects by

$$\mathrm{cr}(T)(\langle n\rangle) = \mathrm{coker}(r_* : \bigoplus_{i=1}^{n} T((n-1)_+) \to T(n_+).$$

Theorem 2.1.11 ([Pir00a, Theorem 3.1])**.** *The functors*

$$\mathrm{cr} : \Gamma-\mathrm{mod} \to \Omega-\mathrm{mod},$$
$$\mathrm{cr} : \mathrm{mod}-\Gamma \to \mathrm{mod}-\Omega$$

are equivalences of categories.

We also get that, for $F \in \Gamma-\mathrm{mod}$ and $T \in \mathrm{mod}-\Gamma$,

$$F \otimes_\Gamma T \cong \mathrm{cr}(F) \otimes_\Omega \mathrm{cr}(T)$$

(see [Pir00b, Section 1.10]). Hence, the HKR-theorem (Theorem 2.1.9) shows that

$$\pi_k F(X) \cong \mathrm{cr}(\mathcal{J}_k(H_*X)) \otimes_\Omega \mathrm{cr}(F).$$

This description was used by Pirashvili to prove the result quoted in (2.1) above, and it could potentially be used to give a description of iterated Hochschild homology for smooth algebras.

As a step towards this end, one can show that

$$\mathrm{cr}(\Lambda^{m_1} t \otimes \cdots \mathrm{cr}(\Lambda^{m_n} t)(l_+) = k^{X_l},$$

where

$$X_l = \{1 \leq i_1^{(1)} < \cdots < i_{m_1}^{(1)} \leq l, ..., 1 \leq i_1^{(n)} < \cdots < i_{m_n}^{(n)} \leq l \mid \{i_1^{(1)}, ..., i_{m_1}^{(1)}, ..., i_1^{(n)}, ..., i_{m_n}^{(n)}\} = \{1, ..., l\}\}.$$

2.2 Operations on Hochschild homology

Since the construction $A^{\otimes X}$ is functorial in X, every endomorphism on X gives an operation on $A^{\otimes X}$. In the special case $X = S^1$, there are the so-called Adams operations constructed in [McC93]. These are based on the r-fold covering map of S^1 and *edgewise subdivision* of simplicial sets, and they recover Loday's Adams operations up to a sign. Indeed, McCarthy has shown that for any functor $F : \Gamma \to Sets$, the composition $F \circ S^1$ has a *system of natural operators* generated by the covering maps of the circle, and that this system generates Loday's Adams operations if F is a functor $\Gamma \to Ab$.

In [Bau], Bauer has shown that for any simplicial set X, similar constructions on $S^1 \wedge X$ give a decomposition of $A^{\otimes S^1 \wedge X}$ that recovers Pirashvili's decomposition in the case $X = S^{n-1}$. This was done by defining a Hopf–algebra structure on the complex $A^{\otimes S^1 \wedge X}$, and then using this structure to get Adams operations and eulerian idempotents, from which the decomposition follows (as in [LQ84] and [GS87]). In this section, we will recall McCarthy's and Bauer's constructions and show how they apply to higher Hochschild homology.

We start by defining edgewise subdivisions and systems of natural operators on a simplicial set X, in the sense of [McC93]. This definition has been extended by Bauer to the case $S^1 \wedge X$, and she has shown that there are operators Φ^r on the complex $F(S^1 \wedge X)$ constructed by using covering maps of the circle. This can be used to define the Adams operations on $A^{\otimes S^1 \wedge X}$ and obtain a decomposition of $\pi_r A^{\otimes S^1 \wedge X}$. In the case $X = S^{n-1}$, Bauer has shown that the decomposition recovers Pirashvili's decomposition of $\pi_r A^{\otimes S^n}$ in Proposition 2.1.7 above, because the operators Φ^r act as multiplication by r^j on both the j-th part of Pirashvili's and of Bauer's decompositions. We will use Bauer's constructions to define Adams operations on higher order Hochschild homology, and, in a similar way as for Bauer's decomposition, we will show how they relate to the decomposition obtained in Corollary 2.1.8. Our exposition is based on [Bau] and [McC93].

2.2.1 Bauer's Constructions

Let $\mathrm{sd}_r : \Delta \to \Delta$ be the functor sending $[n]$ to $[n] \amalg \cdots \amalg [n]$, and let X be a simplicial set. The *r-th edgewise subdivision* of X is the composition of functors $X \circ sd_r$, written $\mathrm{sd}_r(X)$.

Let X be a simplicial set. Following [McC93], we call a collection of simplicial maps $\phi^r : \mathrm{sd}_r X \to X$, $r \in \mathbb{N}$, a *natural system of operators* on X if

$$\begin{array}{ccc} sd_{rs}(X) & \xrightarrow{sd_r(\phi^s)} & sd_r(X) \\ & \searrow^{\emptyset^{r_s}} & \downarrow \phi^r \\ & & X \end{array}$$

commutes for all r, s in $\mathbb{N}$.

Let $d_r : \Delta^{n-1} \to \Delta^{rn-1}$ be the diagonal map

$$d_r(u) = u/r \oplus \cdots \oplus u/r.$$

Lemma 2.2.1 ([BHM93]). *For a simplicial set X, the map $1 \times d_r : X_{rn-1} \times \Delta^{n-1} \to X_{rn-1} \times \Delta^{rn-1}$ induces a homeomorphism $D_r : |\,\mathrm{sd}_r X| \cong |X|$ on realizations.*

From a natural system of operators ϕ^r on X, we get a system of operators Φ^r on $|X|$ through the composite

$$\Phi^r : |X| \xrightarrow{D_r^{-1}} |sd_r(X)| \xrightarrow{|\phi^r|} |X|.$$

Based on the diagram

$$\begin{array}{ccc} \mathrm{sd}_{rs} S^1 \wedge X & \xrightarrow{\mathrm{sd}_r(\phi^s)\wedge 1} & \mathrm{sd}_r S^1 \wedge X \\ & \searrow^{\phi^{rs}\wedge 1} & \downarrow{\scriptstyle \phi^r \wedge 1} \\ & & S^1 \wedge X, \end{array}$$

Bauer has extended McCarthy's definition of a natural system of operators on X to natural systems of operators on $S^1 \wedge X$ ([Bau]).

As in McCarthy's case, a natural system of operators gives rise to maps Φ^r through the composite

$$\begin{array}{ccccc} |S^1 \wedge X| & \xrightarrow{\cong} & |S^1| \wedge |X| & & \\ & & \downarrow{\scriptstyle D_r^{-1}\wedge 1} & & \\ & & |sd_r S^1| \wedge |X| & \xrightarrow{|\phi^r|\wedge 1} & |S^1| \wedge |X| \xrightarrow{\cong} |S^1 \wedge X|. \end{array}$$

There is a certain system of natural operators ϕ^r on S^1 so that Φ^r corresponds to the r-fold covering of the circle in $S^1 \wedge X$. The system was constructed in [Bau, Example 3.2], where it was shown that it could be defined as the composition of what she called the *pinch* map $p^r : sd_r S^1 \to S^1 \vee \cdots \vee S^1$ and the *fold* map $+ : S^1 \vee \cdots \vee S^1 \to S^1$.

Bauer has shown that the operators $\Phi^r : |S^1 \wedge X| \to |S^1 \wedge X|$ we get from this system can be extended to operators

$$\Psi^r : |F(S^1 \wedge X)| \to |F(S^1 \wedge X)|$$

([Bau, Theorem 3.3]). Moreover, she has shown that the operators Ψ^r can be defined on the chain level. To do this, she used the following two lemmas.

Lemma 2.2.2 ([McC93, Proposition 3.4]). *Let X be a simplicial abelian group. Then there is a natural chain map*

$$D(r) : C_*X \to C_*(sd_r X)$$

that passes to the normalized complex. Furthermore, for all $r \in \mathbb{N}$, $D(r)$ is a quasi-isomorphism, and $H_*(D(r)) = \pi_*(D_r)^{-1}$.

Lemma 2.2.3 ([Bau, Lemma 4.2]). *There is a natural chain equivalence*

$$D(r) \wedge 1 : C_*F(S^1 \wedge X) \to C_*(F(\mathrm{sd}_r S^1 \wedge X).$$

Theorem 2.2.4 ([Bau, Theorem 4.3]). *If Φ^r is given by*

$$\Phi^r : F(S^1 \wedge X) \xrightarrow{D(r)\wedge 1} F(sd_r S^1 \wedge X) \xrightarrow{F(\phi^r \wedge 1)} F(S^1 \wedge X),$$

then Φ^r is well defined on the chain level. Moreover, these maps Φ^r agree with the maps Ψ^r on homology.

In the case $X = *$ and $F = \mathcal{L}(A)$, McCarthy ([McC93]) has shown that the operators Φ^r, arising from the system in [Bau, Example 3.2], agree with Loday's Adams operations λ^r on the complex $A^{\otimes S^1}$ up to a sign. For an arbitrary X, the operators Φ^r are called the *Adams operations on* $A^{\otimes S^1 \wedge X}$.

In [Bau, Section 5], Bauer shows that the Hopf–algebra structure on the Hochschild complex giving rise to the decomposition can be expressed in terms of the pinch and fold map on the circle, and by using this viewpoint, she was able to extend the Hopf–algebra structure to the chain complex $A^{\otimes S^1 \wedge X}$. Specifically, she shows the following results:

Lemma 2.2.5 ([Bau, Lemma 5.2 and Lemma 5.3])**.** *Let μ_f be the chain map*

$$A^{\otimes S^1} \otimes_A A^{\otimes S^1} \cong A^{\otimes S^1 \vee S^1} \to A^{\otimes S^1}$$

induced by the fold map $S^1 \vee S^1 \to S^1$, and let Δ_p be the chain map

$$A^{\otimes S^1} \overset{\simeq}{\to} A^{\otimes sd_2 S^1} \to A^{\otimes S^1 \vee S^1}$$

induced by the pinch map $sd_2 S^1 \to S^1 \vee S^1$. By letting these be the multiplication and comultiplication, respectively, the Hochschild complex is a commutative graded Hopf algebra over A, up to homotopy. Moreover, this Hopf–algebra structure agrees with the Hopf–algebra structure defined by Loday and Gerstenhaber–Schack on the normalized Hochschild complex.

Theorem 2.2.6 ([Bau, Theorem 5.5])**.** *The total complex of the bicomplex $A^{\otimes S^1 \wedge X}$ is a commutative bigraded Hopf algebra, with multiplication induced by the fold map on S^1 and comultiplication induced by the pinch map on S^1.*

By using this Hopf–algebra structure, one gets eulerian idempotents (see e.g. [Lod89], [Lod97], or [GS87]), and Bauer has shown that the Adams operations satisfy the relation

$$\Phi^r = re_n^{(1)} + \cdots + r^n e_n^{(n)}. \tag{2.2}$$

By e.g. [Lod89], the eulerian idempotents commute with the Hochschild boundary, and this fact leads to the decomposition of $\pi_n HH^{S^1 \wedge X}(A)$.

Theorem 2.2.7 ([Bau, Theorem 6.1])**.** *Let X be any simplicial pointed set. Then*

$$\pi_n A^{\otimes S^1 \wedge X} = \pi_n e_n^{(1)} A^{\otimes S^1 \wedge X} \oplus \cdots \oplus \pi_n e_n^{(n)} A^{\otimes S^1 \wedge X},$$

where $\pi_n e_n^{(n)} A^{\otimes S^1 \wedge X}$ is the n-th homology of the complex $e_n^{(n)} A^{\otimes S^1 \wedge X}$.

The decomposition agrees Pirashvili's decomposition in the case $X = S^{n-1}$ ([Bau, Theorem 7.3]). This can be proved by showing that the Adams operations Φ^r act by multiplication by r^j on the terms $\mathrm{Tor}_p^\Gamma(\mathcal{J}_{jd}(H_* S^d), \mathcal{L}(A))$, as well as on the j-th term in Bauer's decomposition. For Bauer's decomposition, this fact follows directly from Formula 2.2, and for Pirashvili's decomposition Bauer proves the following lemma.

Lemma 2.2.8 ([Bau, Lemma 7.2])**.** *The map Φ^r acts on $\mathrm{Tor}_p^\Gamma(\mathcal{J}_{dj}(H_* S^n), \mathcal{L}(A))$ by multiplication by r^j.*

Proof. We give Bauer's proof of this lemma.

We have $\mathcal{J}(H_*S^d)(n_+) = (H_*S^d)^{\otimes n} = (k \oplus kx)^{\otimes n}$, where we have written $H_*S^d = k \oplus kx$, with $|x| = d$. From Lemma 2.1.6, we have

$$\mathcal{J}_i(H_*S^d)(n_+) = \begin{cases} 0, \text{ if } i \neq jd \\ (kx^{\otimes j})^{\binom{n}{j}}, \text{ if } i = jd. \end{cases}$$

By the definition of Φ^r, it acts on $H_*S^d = k \oplus kx$ by multiplication by r on kx, so it acts as multiplication by r^j on $\mathcal{J}_{jd}(H_*S^d)(n_+)$. This does not depend on n, so the same result holds for $\mathcal{J}_{jd}(H_*S^d)$.

Since Φ^r is defined on the chain level, this completes the proof. □

2.2.2 Operations on Higher Hochschild Homology

We write $\mathbf{r}_n = (r_1, ..., r_n)$ and

$$\operatorname{sd}_{\mathbf{r}_n} T^n := \operatorname{sd}_{r_1} S^1 \times \cdots \times \operatorname{sd}_{r_n} S^1.$$

Analogous to the definition in the previous section, we define a natural system of operators on T^n as a collection of operators $\phi_{\mathbf{r}_n} : sd_{\mathbf{r}_n} T^n \to T^n$ that make the diagram

$$\begin{array}{ccc} \operatorname{sd}_{\mathbf{r}_n\mathbf{s}_n} T^n & \xrightarrow{\operatorname{sd}_{\mathbf{r}_n}(\phi_{\mathbf{s}_n})} & \operatorname{sd}_{\mathbf{r}_n} T^n \\ & \searrow^{\phi_{\mathbf{r}_n\mathbf{s}_n}} & \downarrow \phi_{\mathbf{r}_n} \\ & & T^n \end{array}$$

commute. These operators give rise to operators $\Phi_{\mathbf{r}_n}$ on $|T^n|$ defined as follows:

$$\begin{array}{llll} \Phi_{\mathbf{r}_n} : |T^n| \xrightarrow{\cong} & |S^1| \times \cdots \times |S^1| & & \\ & \downarrow D_{r_1}^{-1} \times \cdots \times D_{r_n}^{-1} & & \\ & |sd_{r_1} S^1| \times \cdots \times |sd_{r_n} S^1| & \xrightarrow{|\phi_{r_1}| \times \cdots \times |\phi_{r_n}|} & |S^1| \times \cdots \times |S^1| \\ & & & \downarrow \cong \\ & & & |T^n|. \end{array}$$

The natural system of operators defined in [Bau, Example 3.2] gives a system of operators Φ^r on $|T^n|$ that corresponds to the coverings of the torus.

Consider the operators

$$D(\mathbf{r}_n) = D(r_1) \times \cdots \times D(r_n) : C_*F(T^n) \to C_*F(\operatorname{sd}_{\mathbf{r}_n} T^n),$$

where $D(r_i)$ is as in Lemma 2.2.2. As in Bauer's case, we get the following result:

Corollary 2.2.9. *If $\Phi^{\mathbf{r}_n}$ is given by*

$$\Phi^{\mathbf{r}_n} : F(T^n) \overset{D(\mathbf{r}_n)}{\to} F(\operatorname{sd}_{\mathbf{r}_n} T^n) \overset{F(\phi^{\mathbf{r}_n})}{\longrightarrow} F(T^n),$$

then $\Phi^{\mathbf{r}_n}$ is defined on the chain level.

Following Bauer, we can get at the relation between the operator $\Phi^{\mathbf{r}_n} = \Phi^{r_1} \otimes \cdots \otimes \Phi^{r_n}$ and the decomposition in Corollary 2.1.8. Similar to Bauer's result as stated in Lemma 2.2.8, we have the following result:

Lemma 2.2.10. *The map $\Phi^{\mathbf{r}_n}$ acts as multiplication by*

$$\bigoplus_{m_1+\cdots+m_n} r_1^{m_1} r_2^{m_2} \cdots r_n^{m_n}$$

on

$$\bigoplus_{p+q=n} \operatorname{Tor}_p^{\Gamma}(\mathcal{J}_q(H_* T^n), \mathcal{L}(A)).$$

Proof. From the proof of Corollary 2.1.8, we have

$$\mathcal{J}_q(H_* T^n) \cong \bigoplus_{m_1+\cdots m_n = q} \mathcal{J}_{m_1}(H_* S^1) \otimes \cdots \otimes \mathcal{J}_{m_n}(H_* S^1).$$

So the result can be proven exactly as in Lemma 2.2.8.

$$\begin{array}{ccc} \oplus\mathcal{J}_{m_1}(H_*S^1) \otimes \cdots \otimes \mathcal{J}_{m_n}(H_*S^1) & \xrightarrow{\oplus r_1^{m_1} r_2^{m_2} \cdots r_n^{m_n}} & \oplus\mathcal{J}_{m_1}(H_*S^1) \otimes \cdots \otimes \mathcal{J}_{m_n}(H_*S^1) \\ \downarrow & & \downarrow \\ \mathcal{J}_q(H_*(T^n)) & \longrightarrow & \mathcal{J}_q(H_*(T^n)). \end{array}$$

□

2.2.3 Further Operations

As mentioned in the introduction to this section, every endomorphism on X gives an operation on $A^{\otimes X}$ since the construction is functorial in X. Thus, all operators on T^n give operators on $A^{\otimes T^n}$, and since

$$\mathcal{E}\mathrm{nd}(T^n) = \mathcal{E}\mathrm{nd}(B\mathbb{Z}^n) = \mathcal{E}\mathrm{nd}(\mathbb{Z}^n) = M_n\mathbb{Z}$$

in the homotopy category, we can write the operations as $(n \times n)$-matrices with entries in $\mathbb{Z}$. The Adams operations defined in the previous section then correspond to the matrices

$$\begin{pmatrix} r_1 & 0 & \cdots & 0 & \cdots \\ 0 & r_2 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & r_n \end{pmatrix}.$$

As another example, the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

will give an operation τ on $A^{\otimes(S^1 \times S^1)}$ that interchanges the two circles. This follows from the simplicial model we have used for S^1 (Section 1.1).

Example 2.2.11. Let $A = k[t]$, with $|t| = 0$. From Remark 1.3.10, we have

$$k[t]^{\otimes T^2} \cong \bigoplus_{n \geq 0} t^n k[\mathrm{Sp}^n\, T^2],$$

and that

$$\pi_*(k[t])^{\otimes T^2} \cong H_* T^2 \otimes \left(\bigoplus_{n \geq 0} t^n k[\xi]/\xi^n \right) \cong k[t, \epsilon, \eta, x]/\epsilon^2 = \eta^2 = 0.$$

Hence, the effect of applying an operation given by a matrix $B \in M_2\mathbb{Z}$ to $\pi_*(k[t])^{\otimes T^2}$ is determined by the effect it has on H_*T^2. Therefore, a (2×2)-matrix B

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

operates on $\pi_*(k[t])^{\otimes T^2}$ by

$$\begin{aligned} \epsilon &\mapsto a\epsilon, \\ \eta &\mapsto d\eta, \\ x &\mapsto \det(B)x. \end{aligned}$$

For example, the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

interchanges ϵ and η, and it sends x to $-x$ in $\pi_*(k[t])^{\otimes T^2}$. As another example, the matrix

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

sends ϵ to $r\epsilon$, η to $s\eta$, and x to rsx.

More generally, if A is a CDGA with differential 0, then the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

operates on $HH(HH(A, 0)) \cong (\Lambda(V \oplus d_1 V \oplus d_2 V \oplus d_2 d_1 V), 0)$ by interchanging d_1 and d_2, so it sends $(\Lambda(V \oplus d_1 V \oplus d_2 V \oplus d_2 d_1 V), 0)$ to $(\Lambda(V \oplus d_2 V \oplus d_1 V \oplus -d_1 d_2 V), 0)$.

Chapter 3

Cyclic Homology

Connes has observed that the realization of the Hochschild complex has a circle action, and that the cyclic homology of the algebra A is the homotopy of the homotopy orbits under this action.

Theorem. *There is a canonical isomorphism*

$$HC_q A \cong \pi_q |A^{\otimes S^1}|_{hS^1}.$$

This is a special case of a more general result that can be explained as follows. Let M be a *cyclic* simplicial abelian group. Then, according to [DHK85], the realization of the underlying simplicial abelian group has an action by the circle group $\mathbb{T}^1$ (Section 3.2.2). The result then says that the *cyclic homology groups* of M agree with the homotopy groups of the *homotopy orbits* of the realization under the action of the circle group ([Jon87]). We will define n-th order cyclic homology, written $HC_*^{[n]}(A)$, and generalize the above theorem by showing that

$$HC_q^{[n]}(A) \cong \pi_q |A^{\otimes T^n}|_{hT^n}.$$

That is, n-th order cyclic homology will turn out to be the homotopy orbits of the n-th order Hochschild complex under the action of the n-torus.

We start out by defining cyclic objects and cyclic homology of these. In Section 3.2, we will give a proof of the above results.

3.1 Cyclic Objects and Cyclic Homology

A *cyclic object* X in a category $\mathcal{C}$ is a simplicial object in $\mathcal{C}$, that is, a functor $\Delta^o \to \mathcal{C}$, with some additional structure. This structure can be encoded in a category Λ, called *Connes's cyclic category*, so that a cyclic object in $\mathcal{C}$ is a functor $\Lambda^o \to \mathcal{C}$.

Definition 3.1.1. *The category Λ, called Connes's cyclic category, is the category with the same elements $[n]$ as in Δ, and generating maps*

$$\begin{aligned} \delta^i : \ [n] &\to [n-1], \quad 0 \le i \le n, \ n > 0, \\ \sigma^i : \ [n] &\to [n+1], \quad 0 \le i \le n+1. \end{aligned}$$

subject to the same relations as in Δ, but with an additional map $\tau^n : [n] \to [n]$ satisfying the cyclic relations

$$\begin{array}{rcll} \tau^n\delta^i & = & \delta^{i-1}\tau^{n-1}, \quad \textit{for} \quad 1 \leq i \leq n, & \tau^n\delta^0 = \delta^n, \\ \tau^n\sigma^i & = & \sigma^{i-1}\tau^{n+1}, \quad \textit{for} \quad 1 \leq i \leq n, & \tau^n\sigma^0 = \sigma^n(\tau^{n+1})^2, \\ (\tau^n)^{n+1} & = & id. & \end{array}$$

Remark 3.1.2. The category Λ contains Δ as a subcategory, and we write $j : \Delta \to \Lambda$ for the inclusion.

Functors $X : \Lambda^o \to \mathcal{C}$ are called *cyclic objects of* $\mathcal{C}$, and they form a category written as $\mathcal{C}^c$. As for Δ^o, we denote the generating maps of Λ^o by δ_i, σ_i, and τ_n, and the maps induced in $\mathcal{C}$ by a functor X are denoted d_i, s_i, and t_n, respectively.

Analogous to the standard simplicial sets in Δ there are cyclic sets Λ^n called the *standard cyclic sets* ([DHK85, Section 2]). These are defined by

$$\Lambda^n[k] = \hom_{\Lambda^o}([k], [n]),$$

and they have similar properties as the standard simplicial sets Δ^n. We get a natural Yoneda-isomorphism

$$\hom_{\mathcal{S}^c}(\Lambda[n], X) \cong \hom_{\mathcal{S}}(\Delta[n], X) \cong X_n.$$

Cyclic homology of modules. Let X be a cyclic object in the category k-mod of k-modules. Then there is an associated bicomplex

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \\ \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\ X_2 & \xleftarrow{1-t} & X_2 & \xleftarrow{N} & X_2 & \xleftarrow{1-t} & X_2 & \xleftarrow{N} & \\ \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\ X_1 & \xleftarrow{1-t} & X_1 & \xleftarrow{N} & X_1 & \xleftarrow{1-t} & X_1 & \xleftarrow{N} & \\ \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\ X_0 & \xleftarrow{1-t} & X_0 & \xleftarrow{N} & X_0 & \xleftarrow{1-t} & X_0 & \xleftarrow{N} & , \end{array}$$

where $b = \Sigma_{i=1}^n d_i$, $b' = \sum_{i=1}^{n-1} d_i$, $N = \Sigma_{i=0}^n t^i$, and the homology groups of this bicomplex are called the cyclic homology groups of X ([Lod97]).

Definition 3.1.3. *Let X be a cyclic object in the category of k-modules. The homology of the bicomplex associated as above is called the* cyclic homology of X, *and written as $HC_*(X)$.*

It can be shown that the columns involving b' of this complex are exact, and that we therefore get the same homology from the bicomplex

$$\begin{array}{ccccc} L_2 & \xleftarrow{B} & L_1 & \xleftarrow{B} & L_0 \\ \downarrow b & & \downarrow b & & \\ L_1 & \xleftarrow{B} & L_0 & & \\ \downarrow b & & & & \\ L_0. & & & & \end{array}$$

Here $B = (-1)^{n+1}(1 - t_{n+1})sN$, where $s = s_{n+1} = (-1)^{n+1}t_{n+1}s_n$ (see [Lod97]). This complex is called the *Bb-complex* associated to X, and we denote it by $\mathcal{B}(X)$.

Remark 3.1.4. We observe that we have a short exact sequence

$$0 \to X \to \mathcal{B}(X) \to \mathcal{B}(X)[-2] \to 0,$$

where $\mathcal{B}(X)[-2]$ is the Bb-complex shifted 2 places to the right and X is the Moore complex of X. The resulting long exact sequence is called *Connes's exact sequence*:

$$\cdots \longrightarrow \pi_n H(X) \longrightarrow \pi_n HC(X) \longrightarrow \pi_{n-2} HC(X) \longrightarrow \pi_{n-1} H(X) \longrightarrow \cdots.$$

If we filter the Bb-complex by columns, we get the following spectral sequence:

Lemma 3.1.5 ([LQ84, Theorem 1.9])**.** *There is a spectral sequence abutting to $HC_n(X)$, with $E^1_{pq} = \pi_{q-p}L$, and with $d^1 : \pi_{q-p}X \to \pi_{q-p+1}L$ induced by Connes's operator B.*

Example 3.1.6. The simplicial module given by

$$HH(A)_n = A^{\otimes n+1}$$

has a cyclic structure, given by

$$t(a_0, a_1, ..., a_n) = (a_n a_0, ..., a_{n-1}).$$

This can be shown to make $HH(A)$ into a cyclic module, and the cyclic homology of $HH(A)$ is called the *cyclic homology of the algebra A*, and it is denoted by $HC_*(A)$ ([Con85]).

Cyclic homology of functors. Let $\mathcal{F}$ be the category of finite sets and set maps. A skeleton for this category is given by the objects $\underline{n} = \{0, ..., n\}$. The simplicial circle $S^1 : \Delta^o \to \mathcal{F}$ has a cyclic structure, making it a cyclic object in $\mathcal{F}$ (see [Lod97, Section 6.4]). So, as in the Hochschild homology case, we can associate cyclic homology groups to any functor $F : \mathcal{F} \to$ k-mod by composition with S^1.

Definition 3.1.7. *Let $F : \mathcal{F} \to$ k-mod be a functor. The cyclic homology associated to the cyclic k-module*

$$\Lambda^o \xrightarrow{S^1} \mathcal{F} \xrightarrow{F} \text{k-mod}$$

is called the cyclic homology of the functor F*, and written as $HC_*(F)$.*

The cyclic homology of F is the homology of the double complex

$$\begin{array}{ccccc} F(\underline{2}) & \xleftarrow{B} & F(\underline{1}) & \xleftarrow{B} & F(\underline{0}) \\ {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\ F(\underline{1}) & \xleftarrow{B} & F(\underline{0}) & & \\ {\scriptstyle b}\downarrow & & & & \\ F(\underline{0}). & & & & \end{array}$$

Writing $HH(F)$ for the Hochschild complex associated to F, we can write the above double complex as

$$HH(F) \leftarrow \Sigma HH(F) \leftarrow \Sigma^2 HH(F) \leftarrow \cdots,$$

where $\Sigma^n HH(F)$ is the Hochschild complex shifted by n.

Similarly as for the category Γ of finite pointed sets (see 2.1), there are functor categories $\mathcal{F}-$ mod and mod $-\mathcal{F}$. There are projective generators,

$$\mathcal{F}^n := k[\mathrm{Hom}_{\mathcal{F}}(-, \underline{n})],$$
$$\mathcal{F}_n := k[\mathrm{Hom}_{\mathcal{F}}(\underline{n}, -)]$$

of these two categories, and a bifunctor $- \otimes_{\mathcal{F}} -$ such that

$$F(\underline{n}) \cong F \otimes_{\mathcal{F}} \mathcal{F}^n$$
$$F(\underline{n}) \cong \mathcal{F}_n \otimes_{\mathcal{F}} F.$$

Therefore, to calculate the cyclic homology of a functor $F : \mathcal{F} \to$ k-mod, we can calculate the homology of the bicomplex $L^{[1]}$ given by

$$\begin{array}{ccccc} \mathcal{F}_2 & \longleftarrow & \mathcal{F}_1 & \longleftarrow & \mathcal{F}_0 \\ \downarrow & & \downarrow & & \\ \mathcal{F}_1 & \longleftarrow & \mathcal{F}_0 & & \\ \downarrow & & & & \\ \mathcal{F}_0 & & & & \end{array}$$

tensored with F over $\mathcal{F}$.

We have

$$\mathcal{F}_n = k[\mathrm{hom}_{\mathcal{F}}(-, \underline{n})],$$

so the homology of $L^{[1]}(\underline{m})$ is the cyclic homology of $k[S^1 \times \cdots \times S^1]$ (($m+1$) times).

Example 3.1.8. If we let the functor F be the Loday functor $\mathcal{L}(A)$, we get exactly the cyclic homology of the algebra A in the sense of Connes (see Example 3.1.6).

3.1.1 Higher Order Cyclic Homology

The n-torus $T^n = S^1 \times \cdots \times S^1$ is an *n-cyclic* object in $\mathcal{F}$, meaning that it is a functor

$$T^n : \Lambda^o \times \cdots \times \Lambda^o \to \mathcal{F}.$$

To any functor $\Lambda^o \times \cdots \times \Lambda^o \to$ k-mod we can associate a $2n$ complex by using the n simplicial and the n cyclic directions. As an example, the 4-complex associated to the composition

$$\Lambda^o \times \cdots \times \Lambda^o \xrightarrow{T^n} \mathcal{F} \xrightarrow{F} \text{k-mod}$$

is

$$\begin{array}{ccccc}
\Sigma^2 HH^{[2]}(F) & \xleftarrow{B_2} & \Sigma^3 HH^{[2]}(F) & \xleftarrow{B_2} & \Sigma^4 HH^{[2]}(F) \\
\downarrow B_1 & & \downarrow B_1 & & \downarrow B_1 \\
\Sigma HH^{[2]}(F) & \xleftarrow{B_2} & \Sigma^2 HH^{[2]}(F) & \xleftarrow{B_2} & \Sigma^3 HH^{[2]}(F) \\
\downarrow B_1 & & \downarrow B_1 & & \downarrow B_1 \\
HH^{[2]}(F) & \xleftarrow{B_2} & \Sigma HH^{[2]}(F) & \xleftarrow{B_2} & \Sigma^2 HH^{[2]}(F).
\end{array}$$

Definition 3.1.9. *Let $F : \mathcal{F} \to$ k-mod be a functor. The n-th* order cyclic homology *of F is the homology of the $2n$-complex we get from the composition*

$$\Lambda^o \times \cdots \times \Lambda^o \xrightarrow{T^n} \mathcal{F} \xrightarrow{F} \text{k-mod}\,.$$

We write $HC^{[n]}_*(F)$ for this homology.

Remark 3.1.10. The complexes $HH^{[n]}(F)$ are thought of as n-complexes. That is, we consider $F(T^n)$ as an n-simplicial object of k-mod, see Remark 1.2.3. The Hochschild boundaries we get from the different simplicial directions are denoted by $b_1, ..., b_n$.

We have

$$F(m_1 \times \cdots \times m_n) \cong (\mathcal{F}_{m_1} \otimes \cdots \otimes \mathcal{F}_{m_n}) \otimes_{\mathcal{F}} F,$$

so we can calculate this homology by finding the homology of a $2n$-complex $L^{[n]}$. The complex $L^{[2]}$ is the 4-complex

$$\begin{array}{ccccc}
\Sigma^2 \mathcal{F}^{[2]} & \xleftarrow{B_2} & \Sigma^3 \mathcal{F}^{[2]} & \xleftarrow{B_2} & \Sigma^4 \mathcal{F}^{[2]} \\
\downarrow B_1 & & \downarrow B_1 & & \downarrow B_1 \\
\Sigma \mathcal{F}^{[2]} & \xleftarrow{B_2} & \Sigma^2 \mathcal{F}^{[2]} & \xleftarrow{B_2} & \Sigma^3 \mathcal{F}^{[2]} \\
\downarrow B_1 & & \downarrow B_1 & & \downarrow B_1 \\
\mathcal{F}^{[2]} & \xleftarrow{B_2} & \Sigma \mathcal{F}^{[2]} & \xleftarrow{B_2} & \Sigma^2 \mathcal{F}^{[2]}.
\end{array}$$

The $(i_1, ..., i_n)$th entry in the n-complex $L^{[n]}$ is

$$\mathcal{F}^{[n]}_{i_1,...,i_n} = k[\hom_{\mathcal{F}}(i_1, -)] \otimes \cdots \otimes k[\hom_{\mathcal{F}}(i_n, -)],$$

so the homology of $L^{[n]}(\underline{m})$ is the n-th cyclic homology of the cyclic module $k[T^n \times \cdots \times T^n]$ (($m+1$)-times). This will be important in Chapter 4.

The cyclic homology of an algebra A is now defined as follows:

Definition 3.1.11. *Let A be a k-algebra. The higher order cyclic homology associated to the composition*

$$\Lambda^o \times \cdots \times \Lambda^o \xrightarrow{T^n} \mathcal{F} \xrightarrow{\mathcal{L}(A)} \text{k-mod}$$

is called the n-th order cyclic homology of the algebra A, and is denoted by $HC_*^{[n]}(A)$.

3.2 From Hochschild Homology to Cyclic Homology

We now turn to a different description of cyclic homology, namely as the homotopy orbits of Hochschild homology under cyclic actions.

3.2.1 G-Spaces and Homotopy Orbits

Let G be a simplicial group. A *G-space* is a simplicial space $X \in \mathcal{S}$ with a (left) action of G, i.e. a simplicial space X with a map

$$\mu : G \times X \to X,$$

such that the diagrams

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{1 \times \mu} & G \times X \\ {\scriptstyle m \times 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow[\mu]{} & X \end{array}$$

and

$$\begin{array}{ccc} X & & \\ {\scriptstyle i}\downarrow & \searrow^{1_X} & \\ G \times X & \xrightarrow[\mu]{} & X \end{array}$$

commute. Here m is the multiplication in G, and $i(X) = (e, X)$. A simplicial space with a *right* action of G will be called a G^o-space. We write $\mathcal{S}_G$ for the category of G-spaces. This is a *simplicial model category* in the sense of Quillen [Qui67] (see [GJ99, Theorem V.2.3]). The *product* of a G-space X and a G^o-space Y is the space

$$Y \times_G X = Y \times X/((yg, x) \sim (y, gx)).$$

Definition 3.2.1. *Let X be a G-space. The homotopy orbit space X_{hG} of X under the action of G is the space*

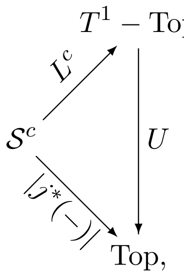
$$X_{hG} = EG \times_G X,$$

where EG is a contractible space with free G-action (see e.g. [GJ99, Definition 3.6]).

3.2.2 Cyclic Sets and T^1-Spaces

According to [DHK85], there is a *Quillen equivalence* (in the sense of [Qui67]) between the category T^1-Top of spaces with an action of T^1 and the category $\mathcal{S}^c$ of cyclic sets. We will start by giving a quick résumé of the results in [DHK85] and show how the results apply to Hochschild homology. See also [DGM, Chapter 3].

As before, $\mathcal{S}^c$ is the category of cyclic objects in the category $\mathcal{S}$. The inclusion $j : \Delta^o \to \Lambda^o$ induces a forgetful functor $j^* : \mathcal{S}^c \to \mathcal{S}$. In [DHK85], Dwyer, Hopkins, and Kan show that there is a functor L^c that fits into the diagram



where $T^1 - \mathrm{Top}$ is the category of topological spaces with an action of the circle group T^1 and U is the forgetful functor. The functor $|j^*(-)|$ is the realization functor taking an object $X \in \mathcal{S}^c$ to the realization $|j^*X|$ of the simplicial set j^*X. From now on, we will suppress the forgetful functor j^* from the realization notation and just write the realization of $X \in \mathcal{S}^c$ as $|X|$. The functor L^c is part of an adjunction-pair

$$L^c : \mathcal{S}^c \rightleftarrows T^1 - \mathrm{Top} : R^c,$$

where R^c is the functor defined by

$$R^c = \hom(L^c\Lambda[-], -) : T^1 - \mathrm{Top} \to \mathcal{S}^c.$$

This pair gives a Quillen equivalence between $T^1 - \mathrm{Top}$ and $\mathcal{S}^c$ ([DHK85, Theorem 4.2]).

In particular, if we have a cyclic module M, we can look at the homotopy orbits of $|M|$ under the action of T^1, and we now have all the ingredients to understand the statement of the theorem quoted in the introduction.

Theorem 3.2.2. *Let M be a cyclic module. Then there is a canonical isomorphism*

$$HC_qM \cong \pi_q|M|_{hS^1}.$$

Following [DGM], we will show how this theorem follows as a special case of a theorem of Jones and a filtration of the homotopy orbits of $|M|$ under the action of T^1. The filtration will be studied in the next section.

Let M be a cyclic module, and write $\varphi : C_*(M) \to C_* \sin|M|$ for the natural chain map. According to the previous results of this section, $|M|$ is a T^1-space, and we write $\mu : T^1 \times |M| \to |M|$ for the T^1-action. Write $J : C_n \sin|M| \to C_{n+1} \sin|M|$ for the map defined by $J(x) = (-1)^{|x|}\mu_*(z * x)$, where $z * x$ is the shuffle product of x with the fundamental 1-cycle in $C_* \sin|S^1|$. In other words, it is the action of T^1 on the realization $|M|$. The theorem of Jones is as follows:

Theorem 3.2.3 ([Jon87, Theorem 4.1])**.** *There is a natural map $h : C_*(M) \to C_* \sin|M|$ that raise the degree by two and satisfies the formula $dh - hb = J\varphi - \varphi B$.*

As in [Jon87], we think of the theorem as saying that the diagram

$$\begin{array}{ccc} C_*(M) & \xrightarrow{\varphi} & C_* \sin|M| \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle J} \\ C_{*+1}(M) & \xrightarrow[\varphi]{} & C_{*+1} \sin|M| \end{array}$$

commutes up to natural chain homotopy, even though this is not exactly the case.

In the next section, we will show how the map J, or the T^1-action, corresponds to the differential in the E^1-sheet of a spectral sequence converging to $\pi_*|M|_{hT^1}$. Then Jones's theorem, together with Lemma 3.1.5, will give us Theorem 3.2.2.

3.2.3 The Bb Complex and the Skeleton Filtration of S^1

We start by showing that the spectral sequence in Lemma 3.1.5 that followed from the filtration of the Bb-complex by columns has E^1-sheet naturally isomorphic to the E^1-sheet of the spectral sequence we get from the skeleton filtration of ET^1, and that the differentials agree on homology. From this, Theorem 3.2.3 will follow.

Filtration of ET^1. A model for the space ET^1 is $\bigcup_{n\geq 0} S^{2n+1}$, where T^1 acts on $S^{2n-1} \subset \mathbb{C}^n$ by complex multiplication in each coordinate. The space ET^1 is filtered via the skeleton filtration

$$* \subset S^1 \subset S^3 \subset \cdots \subset S^{2n-1} \subset S^{2n+1} \subset \cdots \subset S^\infty = ET^1,$$

where the inclusion $S^{2n-1} \subset S^{2n+1}$ is induced by the inclusion of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ as the first coordinates. There is a $\mathbb{T}$-isomorphism

$$S^{2n+1}/S^{2n-1} \overset{\cong}{\to} T^1_+ \wedge S^{2n},$$

given on representatives by $(z_0, ..., z_n) \mapsto (\frac{z_n}{|z_n|} \wedge [z_0, .., z_n])$, where S^{2n} is thought of as $\mathbb{C}P^n/\mathbb{C}P^{n-1}$. From the filtration we get on $ET^1_+ \wedge_{T^1} X$, we get a spectral sequence, called the $\mathbb{T}$-*homotopy orbit spectral sequence*, that has the following description:

Lemma 3.2.4 ([Hes96, Lemma 1.4.2], [DGM, Corollary 3.1.2])**.** *Let X be a T^1-space. The E^2 sheet of the spectral sequence for X_{hT^1} comes from an E^1 sheet with*

$$E^1_{p,q} = \pi_{q-p}X, \; q \geq p \geq 0,$$

and where the differentials

$$d^1_{p,q} : E^1_{p,q} = \pi_{q-p}X \to \pi_{q-p+1}X = E^1_{p-1,q}$$

are induced by $S^1 \wedge X \subset S^1 \wedge X \vee S^0 \wedge X \simeq \mathbb{T}_+ \wedge X \overset{\mu}{\to} X$, where the last map is the $\mathbb{T}$-action.

The E^1-sheet of the spectral sequence can be written as

$$
\begin{array}{lllllll}
 & & \cdots & & \cdots & & \cdots \\
\xleftarrow{d^1_{2,-1}} & \pi_{2-(-1)} & \xleftarrow{d^1_{2,0}} & \pi_{2-0} & \xleftarrow{d^1_{2,1}} & \pi_{2-1} & \xleftarrow{d^1_{2,2}} \\
\xleftarrow{d^1_{1,-1}} & \pi_{1-(-1)} & \xleftarrow{d^1_{1,0}} & \pi_{1-0} & \xleftarrow{d^1_{1,1}} & \pi_{1-1} & \\
\xleftarrow{d^1_{0,-1}} & \pi_{0-(-1)} & \xleftarrow{d^1_{0,0}} & \pi_{0-0} & & & \\
\xleftarrow{d^1_{-1,-1}} & \pi_{-1-(-1)}. & & & & &
\end{array}
$$

Since the map J in Theorem 3.2.3 is the T^1-action, we get the following theorem:

Theorem 3.2.5 ([Jon87])**.** *Let M be a cyclic module. The T^1-action and the B-maps agree on homotopy groups in the sense that the diagram*

$$
\begin{array}{ccc}
\pi_* M & \xrightarrow{\cong} & \pi_*|M| \\
B \big\downarrow & & \big\downarrow d^1 \\
\pi_{*+1} M & \xrightarrow{\cong} & \pi_{*+1}|M|
\end{array}
$$

commutes, where d^1 is the map induced by the T^1-action. The notation $|M|$ means the realization of the simplicial set underlying M, as before.

Theorem 3.2.2 now follows easily.

Proof of Theorem 3.2.2. Let M be a cyclic module. From Lemma 3.1.5 we have the spectral sequence

$$E^1_{pq} = \pi_{q-p} M \Rightarrow HC_{p+q} M,$$

while from Lemma 3.2.4, we have the spectral sequence

$$E^1_{pq} = \pi_{q-p} M \Rightarrow \pi_{p+q}|M|_{hT^1}.$$

By Theorem 3.2.5, the differentials agree, so the result follows. □

3.2.4 Higher Dimensions

We make the following observation: Let X be a $G_1 \times G_2$-space, where G_1 and G_2 are simplicial groups. We have

$$E(G_1 \times G_2) \times_{(G_1 \times G_2)} X \cong EG_2 \times_{G_2} (EG_1 \times_{G_1} X),$$

so the homotopy orbits can be calculated iteratively:

$$X_{h(G_1 \times G_2)} \cong (X_{hG_1})_{hG_2}.$$

This gives an immediate generalization of the result in the previous section to higher dimensions.

We recall from Section 3.2 that the n-th order cyclic homology of an algebra A is the cyclic homology associated to the n-cyclic module

$$\Lambda^o \times \cdots \times \Lambda^o \xrightarrow{T^n} \mathcal{F} \xrightarrow{\mathcal{L}(A)} \text{k-mod}\,.$$

That is, it is the homology of a $2n$-complex whose differentials are $b_1, ..., b_n$ and $B_1, ..., B_n$.

Theorem 3.2.6. *Let A be a k-algebra. Then*

$$HC_*^{[n]}(A) \cong \pi_* |A^{\otimes T^n}|_{hT^n}.$$

Proof. We prove this in the case $n = 2$. The general case follows.

The complex $C_*(A^{\otimes T^2})_{hT^2})$ can be written as $C_* \sin(|HH_1^{[2]}(A)|_{hS^1}|_{hS^1})$, where $HH_1^{[2]}(A)$ denotes the simplicial set $[q] \mapsto |A^{S_q^1 \times S^1}|$. This complex is quasi-isomorphic to

$$\text{Tot}(C_* \sin |HH_1^{[2]}(A)|, b_2, B_2)$$

by the results of the previous section. But

$$\text{Tot}(C_* \sin |HH_1^{[2]}(A)|, b_2, B_2) \simeq \text{Tot}(C_*([q] \mapsto C_*(A^{\otimes S_q^1 \times S^1}, b_1, B_2))),$$

which completes the proof. □

Remark 3.2.7. *Negative* cyclic homology of a cyclic module M, HC_*^-, can be shown to correspond to the homotopy of the S^1-*homotopy fixed points* of $|M|$ under the action of S^1:

$$|M|^{hS^1} = Map_{S^1}(ES^1_+, |M|),$$

so one could consider the *n-th negative cyclic homology* of M as $\pi_* |M|^{hT^n}$.

Chapter 4

Decomposition of Cyclic Homology

Loday ([Lod89]) has shown that for each functor $F \in \text{mod}-\mathcal{F}$, there is a canonical decomposition

$$HC_n(F) \cong \bigoplus_{i=1}^{n} HC_n^{(i)}(F).$$

As for Hochschild homology, Pirashvili generalized this result in [Pir00b] by constructing a degenerate spectral sequence converging to cyclic homology. We will generalize his result by constructing a degenerate spectral sequence converging to higher order cyclic homology.

4.1 Pirashvili's Decomposition

We recall from Section 3.1 that cyclic homology of a functor F can be calculated by finding the homology of the bicomplex $L^{[1]}$:

$$\begin{array}{ccccc}
\mathcal{F}_2 & \longleftarrow & \mathcal{F}_1 & \longleftarrow & \mathcal{F}_0 \\
\downarrow & & \downarrow & & \\
\mathcal{F}_1 & \longleftarrow & \mathcal{F}_0 & & \\
\downarrow & & & & \\
\mathcal{F}_0, & & & &
\end{array}$$

tensored with F over $\mathcal{F}$. More generally, the n-th order cyclic homology of F can be calculated by finding the homology of a $2n$-complex $L^{[n]}$ tensored with F over $\mathcal{F}$.

In [Pir00b], Pirashvili calculates the homology of $L^{[1]}$ as

$$H_i(L^{[1]}) = \Lambda^i \bar{t},$$

where $\bar{t}$ is the right $\mathcal{F}$-module defined below. We will expand Pirashvili's argument to $L^{[n]}$ in Lemma 4.1.3.

Remark 4.1.1. Let $\nu : \mathcal{F} \to \Gamma$ be the functor that adds a disjoint basepoint. Then, for any functor $F : \Gamma \to \text{k-mod}$, we get a functor $\nu^* F : \mathcal{F} \to \text{k-mod}$ by precomposition.

As in [Pir00b, Section 3.4], we let $\mathcal{F}_0 \to \nu^* t$ be the canonical transformation that is defined on $\underline{n}$ as the homomorphism

$$\lambda \mapsto \lambda_{\underline{n}} \in \nu^* t(n_+) = \hom_{Sets}(\underline{n}, k),$$

where $\lambda_{\underline{n}} : S \to k$ is the constant map with value λ. The $\mathcal{F}$-module $\bar{t}$ is now defined as the object fitting in the exact sequence

$$0 \to \mathcal{F}_0 \to \nu^* t \to \bar{t} \to 0.$$

To calculate the homology of $L^{[n]}$, we will need a result from [Lod97]. The graded comodule $k[u]$ in the following theorem is isomorphic to $HC_*(k)$, and $|u| = 2$ (see [Lod97, Section 4.4]):

Theorem 4.1.2 ([Lod97, Proposition 4.4.8])**.** *Let A and A' be two k-algebras. Suppose that*

$$HC_*(A') = k[u] \otimes U_* \oplus V_*,$$

where U_* and V_* are graded trivial $k[u]$-comodules. Then

$$HC_*(A \otimes A') = HC_*(A) \otimes U_* \oplus \pi_* HH(A) \otimes V_*.$$

Lemma 4.1.3. *The homology of the complex $L^{[n]}$ is*

$$H_r(L^{[n]}) \cong \bigoplus_{m_1 + \cdots + m_n = r} \Lambda^{m_1} \bar{t} \otimes \cdots \otimes \Lambda^{m_n} \bar{t}.$$

Proof. From Section 3.1 we have that the homology of $L^{[n]}(\underline{m})$ is the n-th cyclic homology of the cyclic module $k[T^n \times \cdots \times T^n]$ ($(m+1)$ T^n-factors). Write $(T^n)^{m+1} := T^n \times \cdots \times T^n$. By Theorem 3.2.6,

$$HC_*^{[n]}(k[(T^n)^{m+1}]) \cong \pi_*(ET^n \times_{T^n} (T^n)^{m+1}).$$

But $ET^n \times_{T^n} (T^n)^{m+1} \cong ES^1 \times_{S^1} (S^1)^{m+1} \times \cdots \times ES^1 \times_{S^1} (S^1)^{m+1}$, so by Theorem 3.2.6 again,

$$\begin{aligned} HC_*^{[n]}(k[(T^n)^{m+1}]) &\cong \pi_*(ES^1 \times_{S^1} (S^1)^{m+1}) \otimes \cdots \otimes \pi_*(ES^1 \times_{S^1} (S^1)^{m+1}) \\ &\cong HC_*(k[(S^1)^{m+1}]) \otimes \cdots \otimes HC_*(k[(S^1)^{m+1}]). \end{aligned}$$

Thus, it is enough to find the cyclic homology of $k[(S^1)^{m+1}]$. This has been done by Pirashvili in [Pir00b, Section 3.4]. We repeat the argument here.

We have

$$HC_*(k[S^1]) \cong H_*(ES^1 \times_{S^1} S^1) \cong k.$$

Hence, by Theorem 4.1.2, we get that

$$HC_*(k[S^1 \times S^1]) \cong HC_*(k[S^1] \otimes k[S^1]) \cong HH_*(k[S^1]) \cong \Lambda^*(x),$$

where x has degree 1. By iterating this result, we get that

$$HC_*(k[(S^1)^n]) \cong \Lambda^*(x_1, ..., x_n),$$

where each x_i has degree 1.

This shows that

$$H_*(L^{[n]}(\underline{m})) \cong \Lambda^*(x_1^{(1)}, ..., x_m^{(1)}) \otimes \cdots \otimes \Lambda^*(x_1^{(n)}, ..., x_m^{(n)}),$$

all $x_i^{(j)}$ of degree 1, and the result now follows from a similar argument as in Lemma 2.1.6 (see [Pir00b, Section 4.4]).

□

Proposition 4.1.4. *Let F be a left $\mathcal{F}$-module. Then there exists a degenerate spectral sequence*

$$E^2_{pq} = \bigoplus_{m_1+\cdots+m_n=q} Tor_p^{\mathcal{F}}(\Lambda^{m_1}\bar{t} \otimes \cdots \otimes \Lambda^{m_n}\bar{t}, F) \Rightarrow HC^{[n]}_{p+q}(F),$$

with

$$HC^{[n]}_r(F) \cong \bigoplus_{m_1+\cdots+m_n+p=r} Tor_p^{\mathcal{F}}(\Lambda^{m_1}\bar{t} \otimes \cdots \otimes \Lambda^{m_n}\bar{t}, F).$$

Proof. The existence of the spectral sequence follows the calculation in Lemma 4.1.3 and from the $\mathcal{F}$-version of Proposition 2.1.2 (see Remark 2.1.3), by applying the proposition to the total complex of $L^{[n]}$. More precisely, we put $C_* = \operatorname{Tot} L^{[n]}$ and consider the spectral sequence

$$E^2_{pq} = \operatorname{Tor}_p^{\mathcal{F}}(H_q(C_*), F) \Rightarrow H_{p+q}(C_* \otimes_{\mathcal{F}} F) \cong HC^{[n]}_{p+q}(F).$$

The spectral sequence has the desired form by Lemma 4.1.3.

To show that the spectral sequence degenerates, one shows that

$$\operatorname{Ext}^i_{\mathrm{mod}-\mathcal{F}}(\Lambda^{m_1}\bar{t} \otimes \cdots \otimes \Lambda^{m_k}\bar{t}, \Lambda^{m_1}\bar{t} \otimes \cdots \otimes \Lambda^{m_n}\bar{t}) = 0$$

if $k < n$ and $i \geq 2$, but for this, it is enough to show that $\operatorname{Ext}^i_{\mathrm{mod}-\mathcal{F}}(\Lambda^n\bar{t}, \Lambda^m\bar{t}) = 0$, and this is done in the proof of [Pir00b, Theorem 3.5]. □

Index of Notation

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